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UNIVERSITY OF Southampton

FEM Simulations of Non-linear Interactions of Waves and Floating Structures

by

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FEM simulation of nonlinear interactions of waves and floating structures in the context of carbon capture and transportation into ocean spaces

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Governing Equations and Boundary Conditions

- The fluid is assumed to be incompressible and inviscid， and the flow is irrotational.

Fully nonlinear solution

Governing equation: $\nabla^2\phi = 0$

On the free surface:

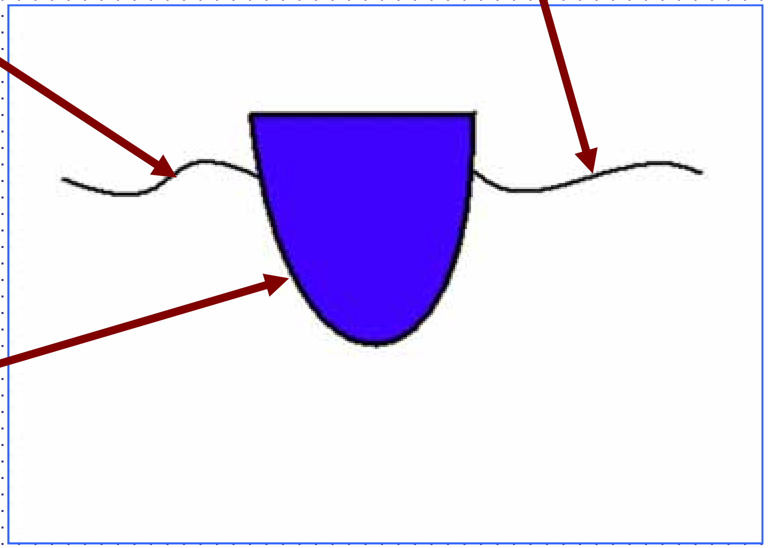
$$\frac{Dx}{Dt} = \frac{\partial\phi}{\partial x}, \quad \frac{Dy}{Dt} = \frac{\partial\phi}{\partial y}, \quad \frac{Dz}{Dt} = \frac{\partial\phi}{\partial z},$$

$$\frac{D\phi}{Dt} = -gz + \frac{1}{2}(\nabla\phi)^2$$

On the body surface:

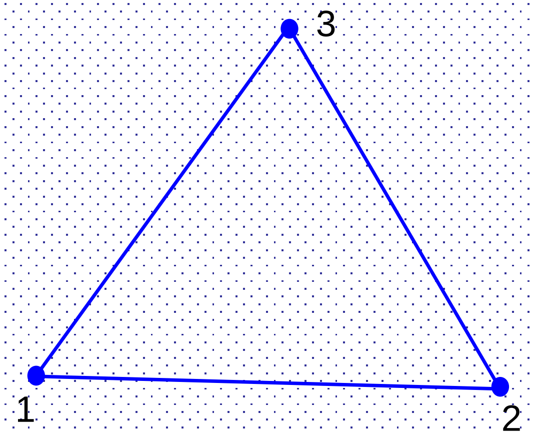
$$\frac{\partial\phi}{\partial n} = \boldsymbol{U}\cdot\boldsymbol{n}$$

Pressure $p=0$

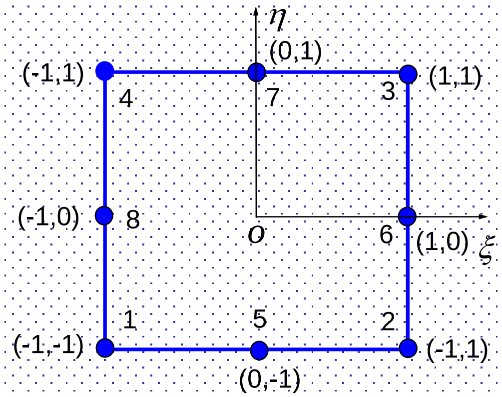


Mesh Generation

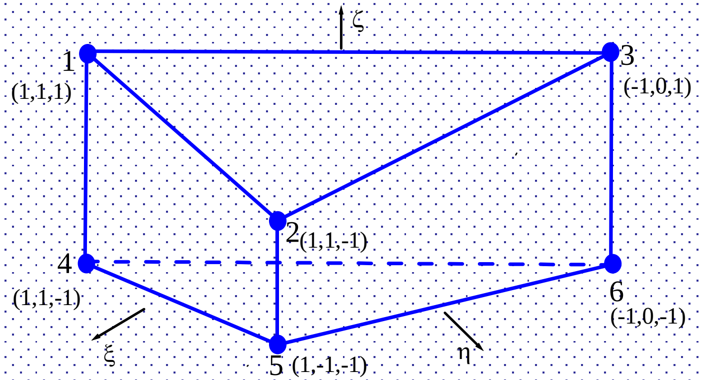
Element types



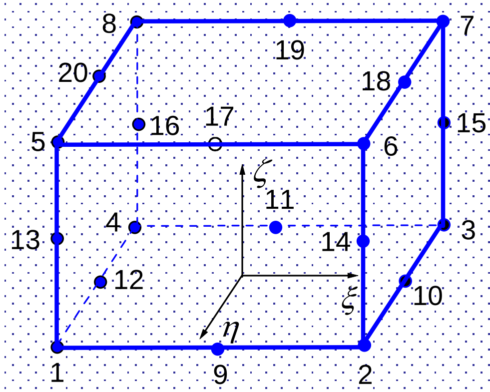
3-node triangular element



8-node quadrilateral element



Triangular 6-node prism element

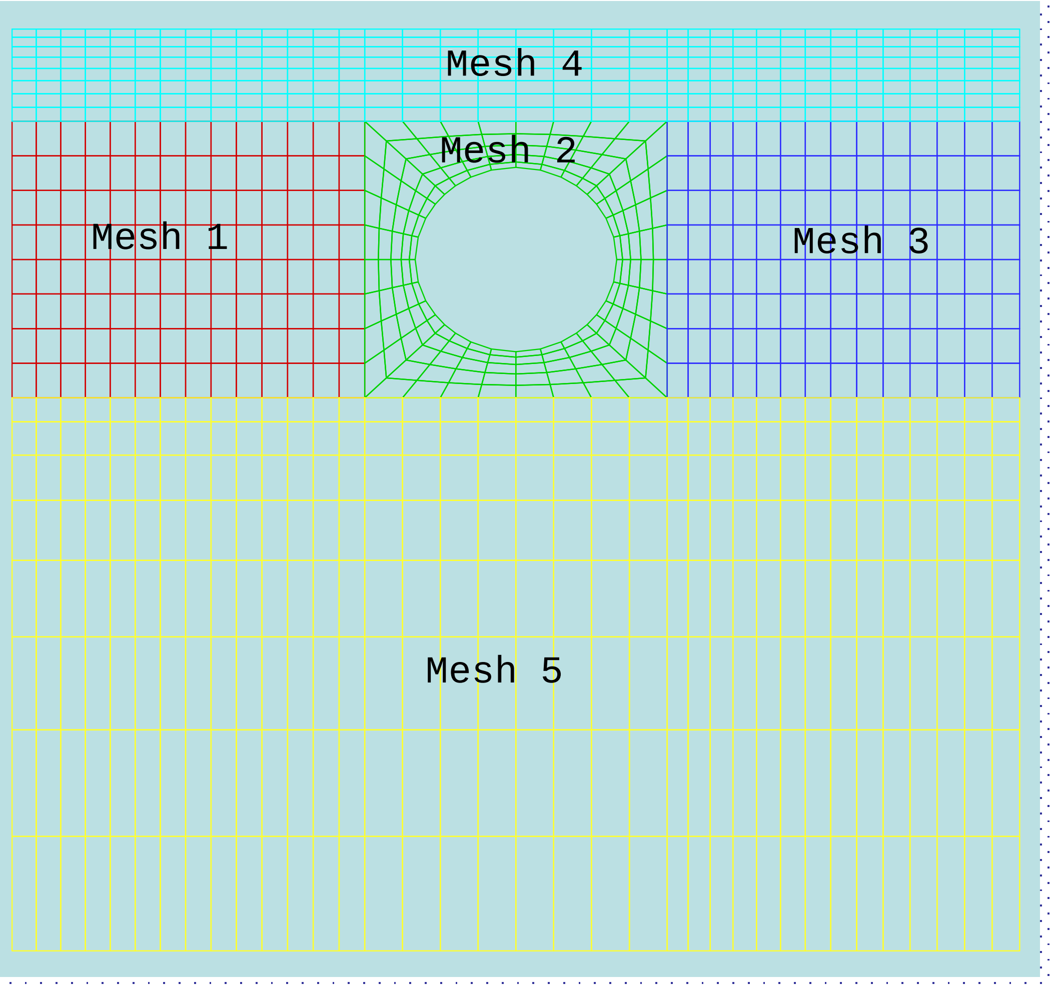


20-node brick element

Multi-block structured meshes

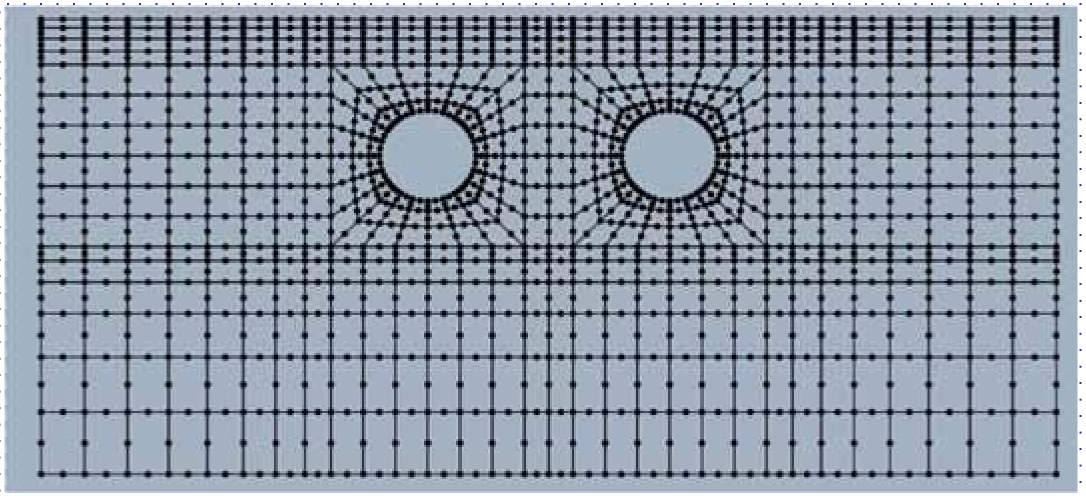
- The fluid domain is subdivided into many simple zones, in which mesh may be generated more easily. A multi-block mesh is then generated by unifying all grids in the simple zones together;
- The numbers of nodes on the boundary of each zone should be provided.

Multi-block structured meshes…

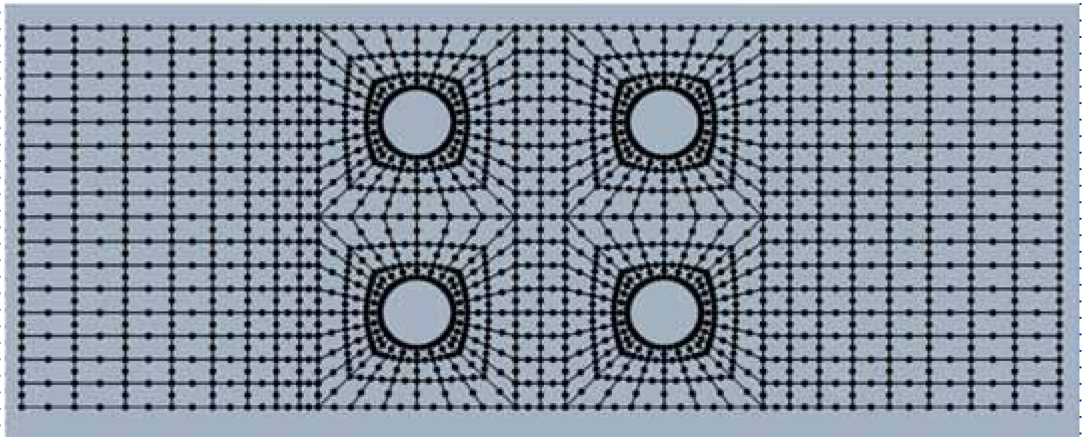


A multi-block mesh for a circular cylinder formed by combining five simple meshes together

Multi-block structured meshes…

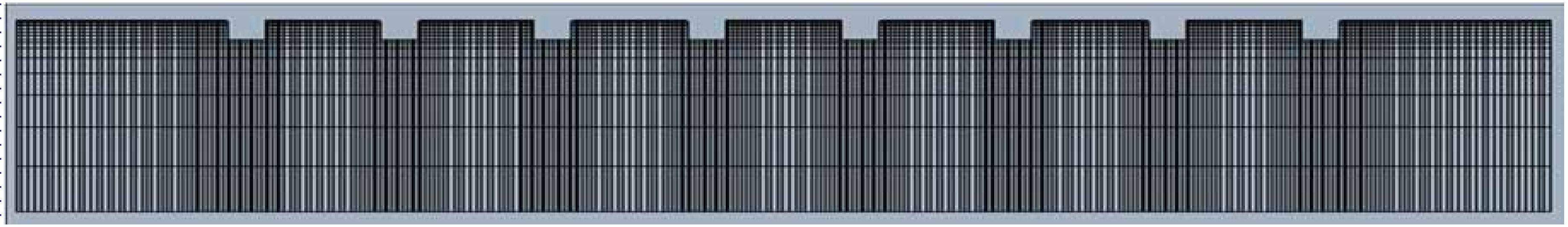


Mesh for two circular cylinders (8-node quadrilateral element)

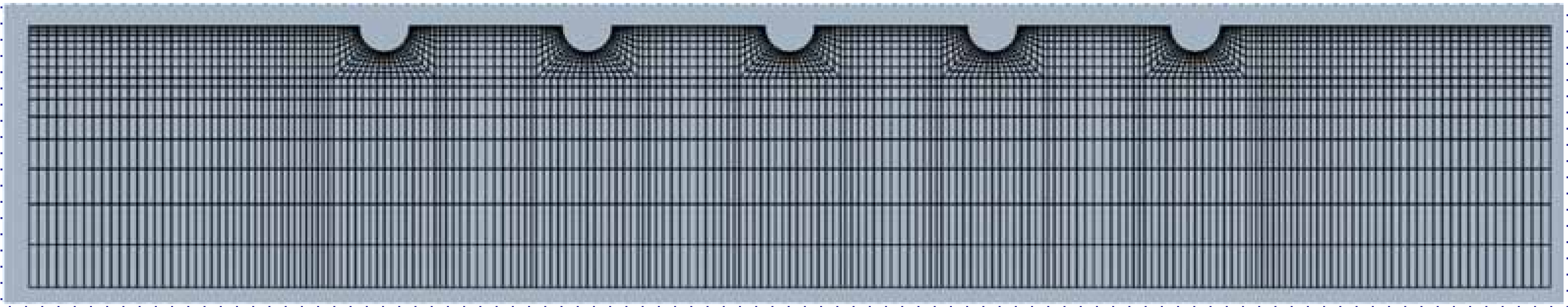


Mesh for four circular cylinders (8-node quadrilateral element)

Multi-block structured meshes…

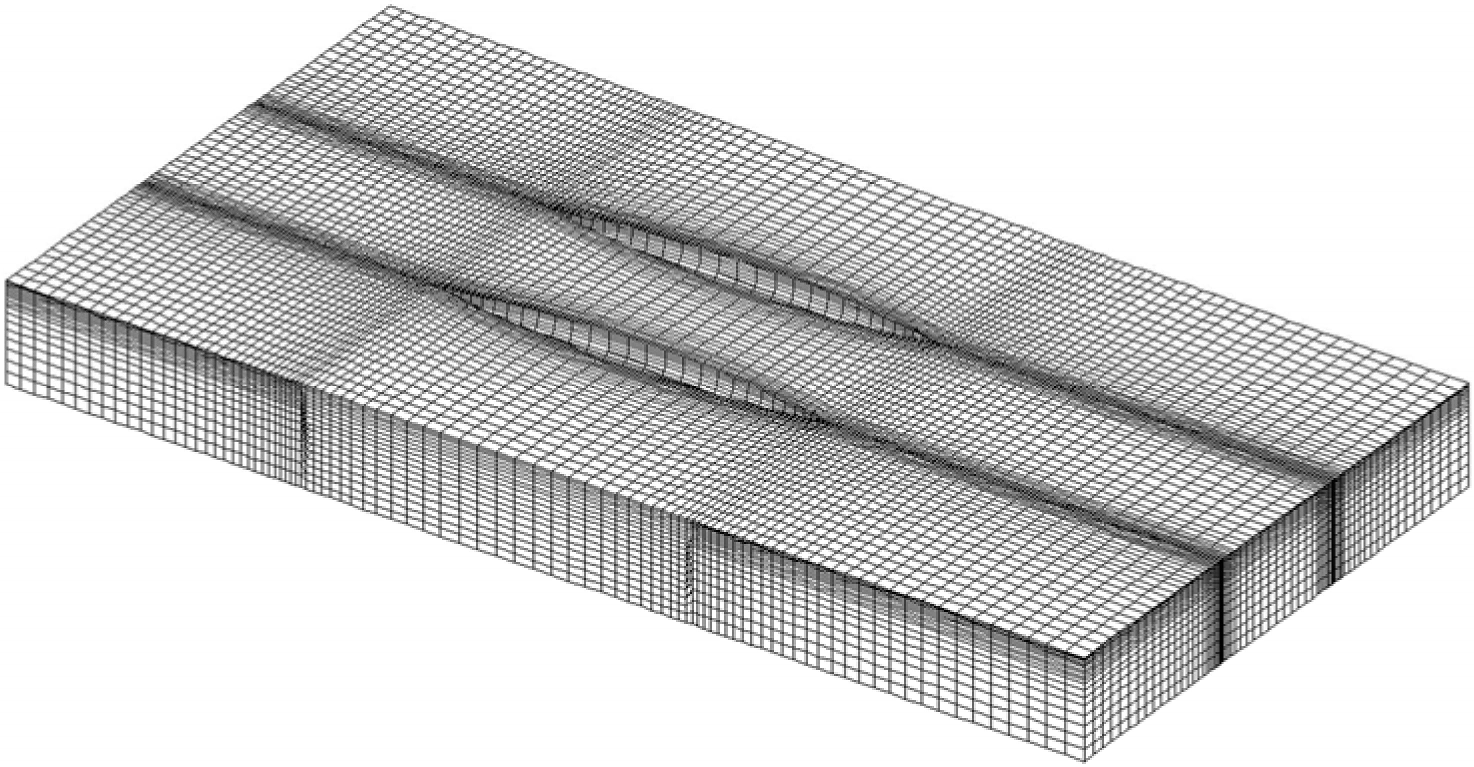


Mesh for eight floating rectangular cylinders (8-node quadrilateral element)



Mesh for five floating semicircular cylinders (8-node quadrilateral element)

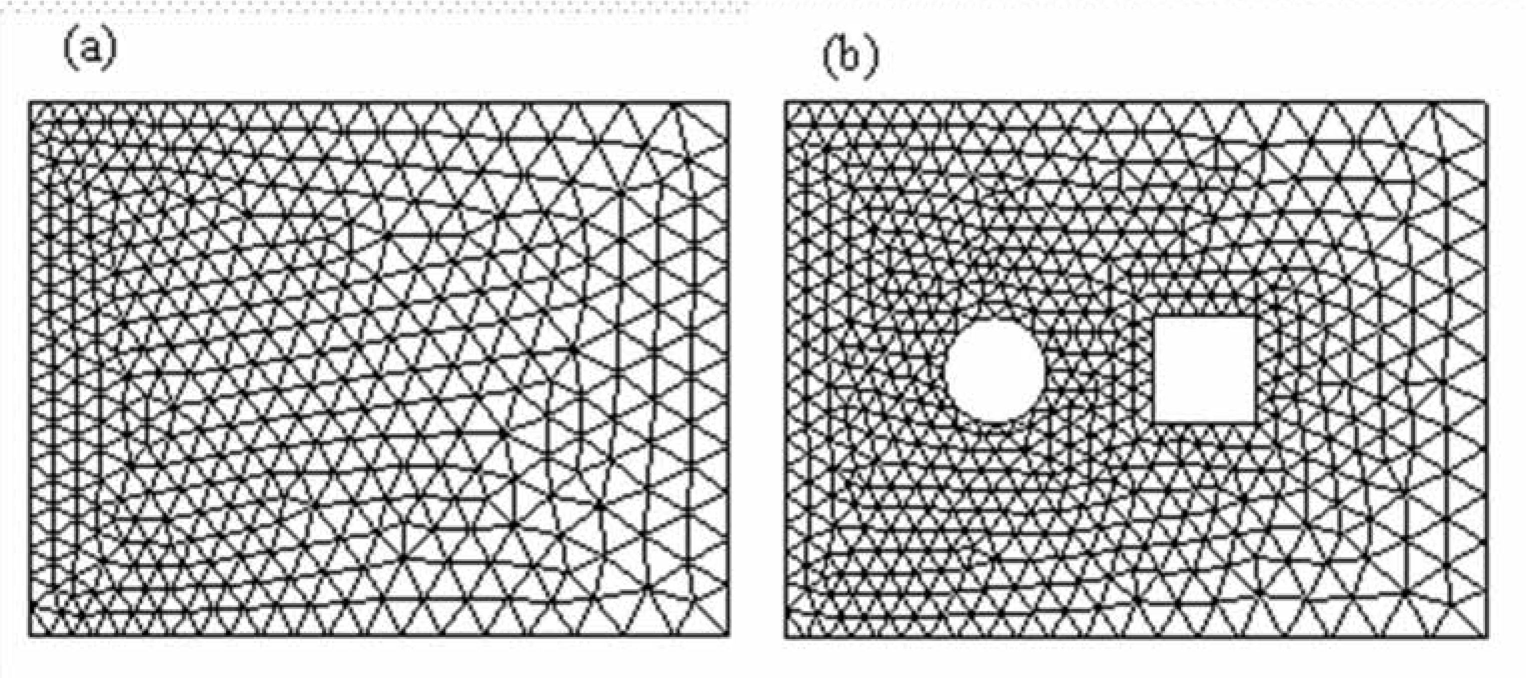
Multi-block Structured Meshes…



3D mesh for a twin Wigley ship (20-node brick element)

Unstructured meshes

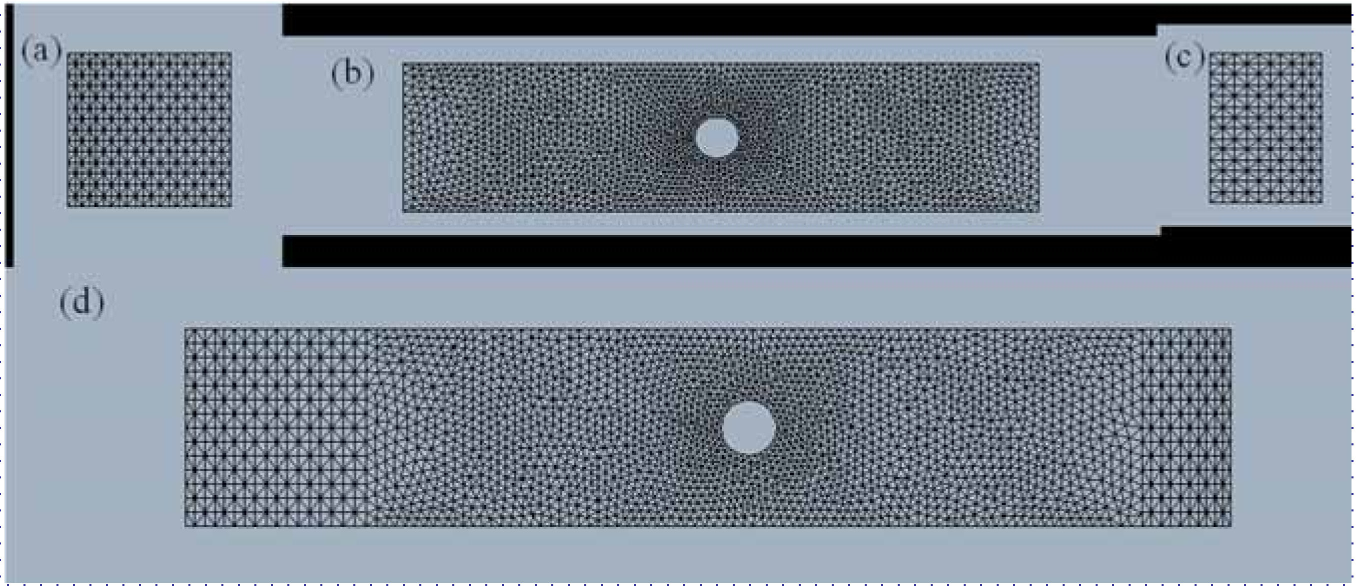
- Delaunay and tri-tree methods: only requires boundary information including nodes and element numbers.



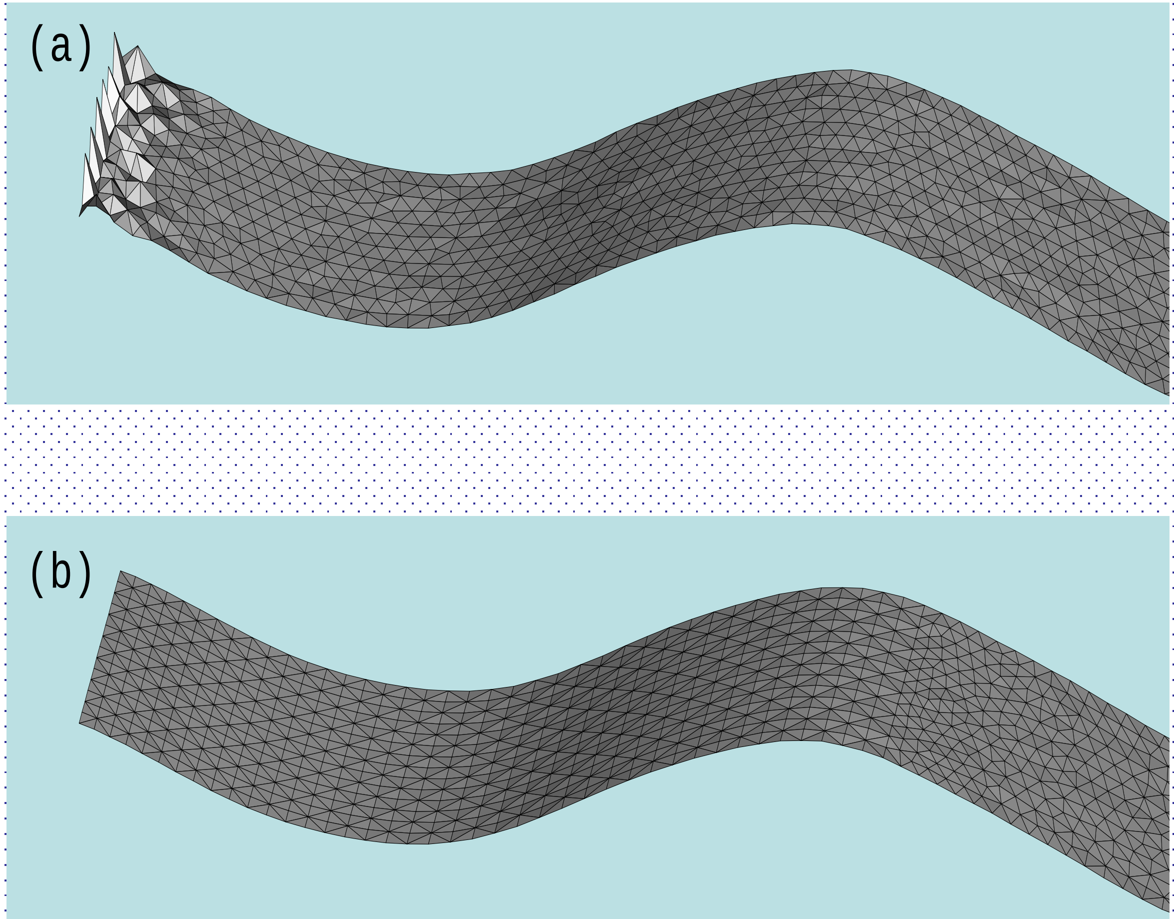
2D unstructured meshes

Unstructured meshes...

- Using hybrid mesh:
    - a) To improve the numerical stability;
    - b) To use 2-D smoothing techniques.

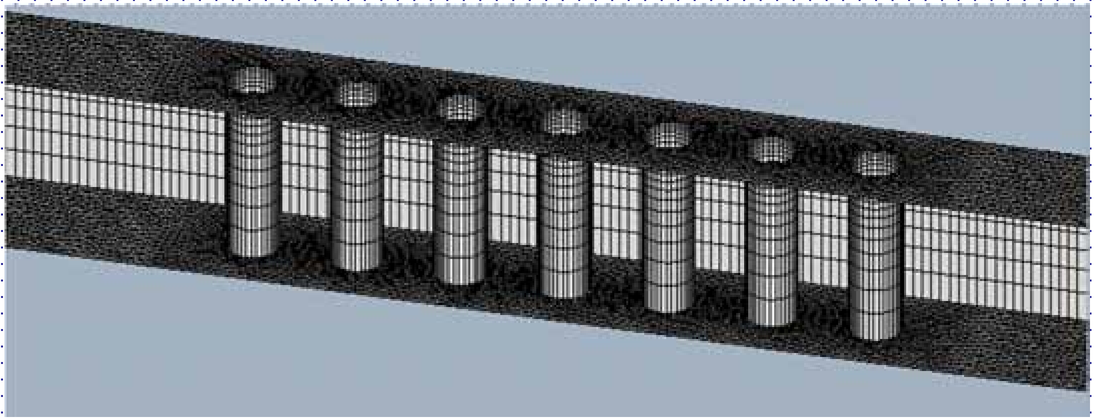


Unstructured Meshes…

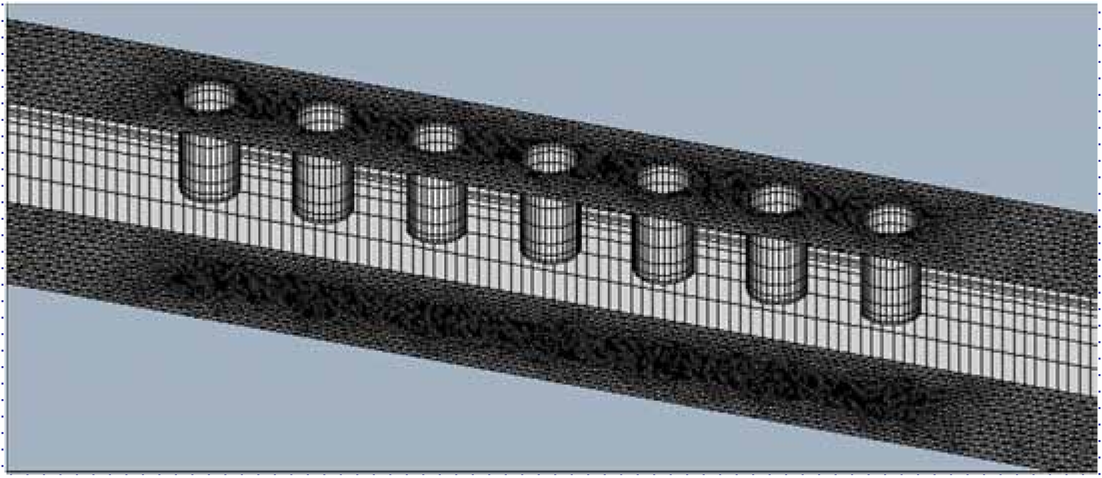


(a) Unstructured mesh without smoothing; (b) Hybrid mesh with smoothing applied within structured mesh (Wu, Ma & Eatcok Taylor 1996, 21st ONR, Trondheim)

Unstructured meshes...



Seven bottom-mounted cylinders (6-node prism element)



Seven truncated cylinders (6-node prism element)
(Wang & Wu, J.Fluids & Strus 2006)

Finite Element Method

Discretisation of equation

- Velocity potential is written in terms of the shape functions:

$$\phi = \sum_J \phi_J N_J(x, y, z)$$

- The Galerkin method:

$$\iiint_\forall \nabla^2 \phi N_I d\forall = 0$$

- Discretised equation after using the Green's identity:

$$\iiint_\forall \nabla N_I \cdot \sum_{\substack{J \\ J \notin S_p}} \phi_J \nabla N_J d\forall = \iint_{S_n} N_I f_n dS - \iiint_\forall \nabla N_I \cdot \sum_{\substack{J \\ J \in S_p}} (f_p)_J \nabla N_J d\forall \quad (I \notin S_p)$$

Matrix form of the equation

$$[A]\{\phi\} = \{B\}$$

$$\{\phi\} = [\phi_1, \phi_2, \phi_3 \cdots \phi_I \cdots]' \left(I \notin S_p\right)$$

$$A_{IJ} = \iiint_{\forall} \nabla N_I \cdot \nabla N_J d\forall \left(I \notin S_p \, \text{and} J \notin S_p\right)$$

$$B_I = \iint_{S_n} N_I f_n dS - \iiint_{\forall} \nabla N_I \sum_{\substack{J \\ J \in S_p}} \left(f_p\right)_J \nabla N_J d\forall \left(I \notin S_p\right)$$

Solve the linear system

- Direct method: the Gaussian elimination or Cholesky method is employed to solve the linear (sparse and) symmetric system and Cuthill-McKee method to optimize bandwidth;
- Iterative method: the conjugate gradient method with a symmetric successive over relaxation (SSOR) preconditioner is used and only nonzero elements in the stiffness matrix are stored.

Calculate first-order (velocity) and second- order derivatives

- Method by differentiating the shape function;
- Difference method;
- Galerkin method (Global projection method).

Method by differentiating the shape functions (Wang, Wu & Drake, 2007, Ocean Eng.)

$$\frac{\partial \phi}{\partial x} = \sum_{i=1}^{n} \phi_j \frac{\partial N_i}{\partial x}, \quad \frac{\partial \phi}{\partial y} = \sum_{i=1}^{n} \phi_i \frac{\partial N_i}{\partial y}, \quad \frac{\partial \phi}{\partial z} = \sum_{i=1}^{n} \phi_i \frac{\partial N_i}{\partial z}$$

where

$$\begin{bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \\ \frac{\partial N_i}{\partial z} \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} & \frac{\partial z}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} & \frac{\partial z}{\partial \eta} \\ \frac{\partial x}{\partial \zeta} & \frac{\partial y}{\partial \zeta} & \frac{\partial z}{\partial \zeta} \end{bmatrix}^{-1} \begin{bmatrix} \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} \\ \frac{\partial N_i}{\partial \zeta} \end{bmatrix}$$

Differential method_

- Employ a cubic polynomial to express the velocity potential along the vertical direction :(Ma, Wu & Eatock Taylor 2001, Int.J. Nume. Meth. in Fluids)

$$\phi = a + bz + cz^2 + dz^3$$

$$w = \frac{\partial \phi}{\partial z} = b + 2cz + 3dz^2$$

$$\left(\vec{u} = (u, v, w) = (\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}) \right)$$

- On the free surface:

$$\left.\begin{aligned} u_i l_x^k + v_i l_y^k &= \frac{\partial \phi}{\partial l^k} - w_i l_z^k \\ u_i l_x^m + v_i l_y^m &= \frac{\partial \phi}{\partial l^m} - w_i l_z^m \end{aligned}\right\}$$

$\frac{\partial \phi}{\partial l^k} = \frac{\phi_{i+k} - \phi_i}{l^k}$ and l_x^k, l_y^k & l_z^k are the components of $\vec{l}^{\,k}$ $(k = 1, 2, \cdots)$, a vector formed by nodes $i + k$ and i.

Galerkin method
(Wu & Eatock Taylor, App. Ocean Res 1994)

$$\iiint_{\forall} (\vec{u} - \nabla \phi) N_i \, d\forall = 0$$

In matrix form:

$$[A]\{\vec{u}\} = [\vec{B}]\{\phi\}$$

where

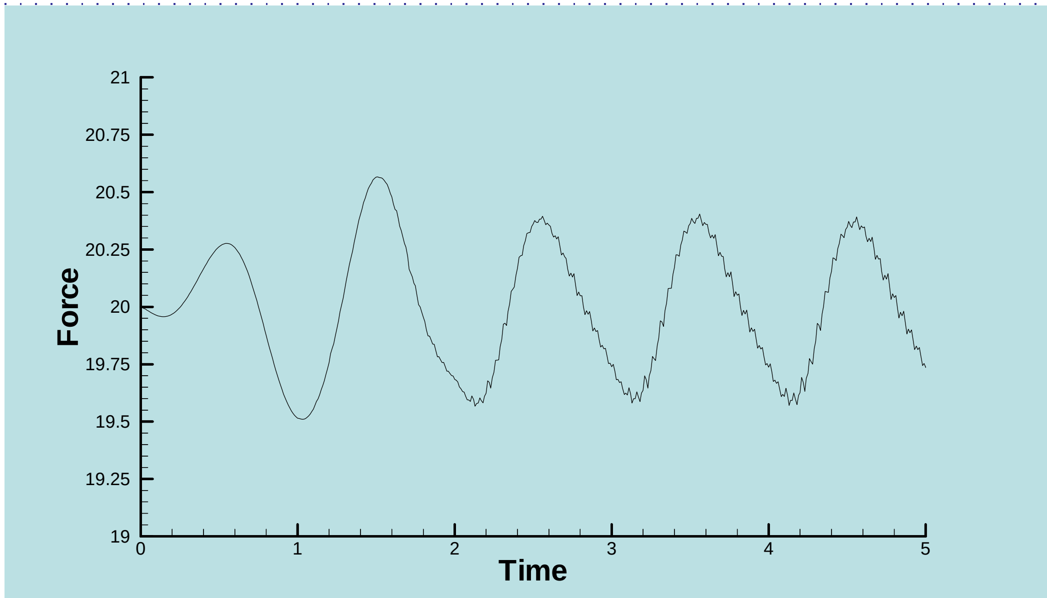
$$A_{ij} = \iiint_{\forall} N_i N_j \, d\forall, \quad \vec{B}_{ij} = \iiint_{\forall} N_i \nabla N_j \, d\forall$$

Equation to Calculate the Force

The Force acting on the body

$$\vec{F} = -\rho \iint_{S_b} \left(\phi_t + \frac{1}{2} |\nabla \phi|^2 + gz \right) \vec{n} ds$$

- Problems with evaluation of $\iint_{S_b} \phi_t \vec{n} ds$: instability and sawteeth behaviour



Evaluation of $\partial\phi/\partial t$:
(Wu, J. Fluids & Strucs. 1998)

In the fluid domain :

$$\nabla^2\phi_t = 0$$

On the free surface:

$$\phi_t = -\frac{1}{2}\nabla\phi\nabla\phi - gz$$

On the moving boundary:

$$\frac{\partial\phi_t}{\partial n} = (\dot{\vec{U}} + \dot{\vec{\Omega}}\times\vec{r}).\vec{n} - \vec{U}.\frac{\partial\nabla\phi}{\partial n} + \vec{\Omega}\cdot\frac{\partial}{\partial n}[\vec{r}\times(\vec{U} - \nabla\phi)]$$

Methods to handle the second order derivatives such as $\partial \nabla \phi / \partial n$ in the moving boundary condition

- To employ high order shape functions to calculate the second order derivatives directly.
- To introduce auxiliary functions to avoid calculating the second order derivatives.

Introduce auxiliary functions $\chi_i (i = 1,2,\cdots,6)$

(Wu & Eatock Taylor 2003, Ocean Eng)

in the fluid domain: $$\nabla^2 \chi_i = 0$$

on the body surface: $$\frac{\partial \chi_i}{\partial n} = n_i$$

on the free surface: $$\chi_i = 0$$

on other boundaries: $$\frac{\partial \chi_i}{\partial n} = 0$$

For multiple structures, the force on *i*-th body:

$$\vec{F}_i = -\iint_{S_b} \{\nabla \chi_i [(\vec{V} + \vec{\Omega} \times \vec{r}) \cdot \vec{n}][\nabla \phi - (\vec{V} + \vec{\Omega} \times \vec{r})] + \chi_i (\vec{\Omega} \times \vec{V}) \cdot \vec{n}\} ds$$

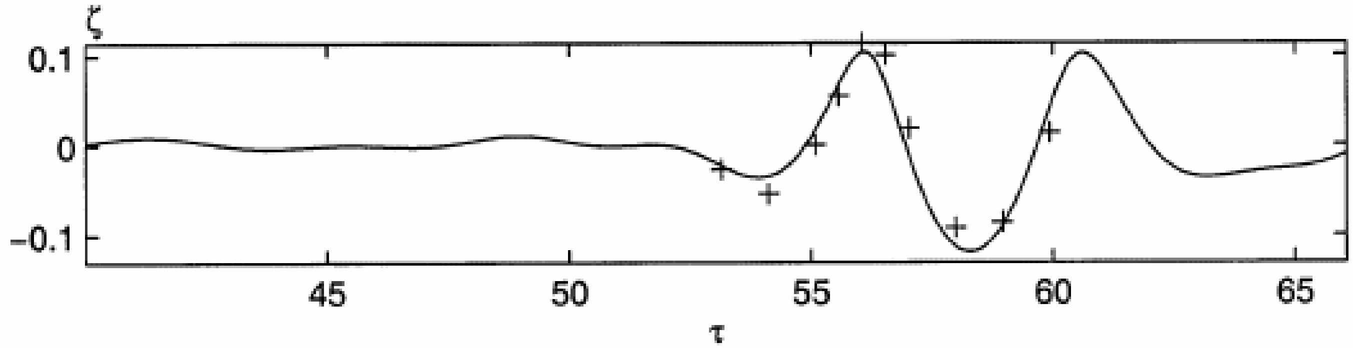
$$- \iint_{S_f + S_b} (\frac{1}{2} \nabla \phi \cdot \nabla \phi + gz) \frac{\partial \chi_i}{\partial n} ds - \sum_{j=1}^{6} C_{ij} A_j \qquad (i = 1, 2, \cdots, 6)$$

Numerical Examples

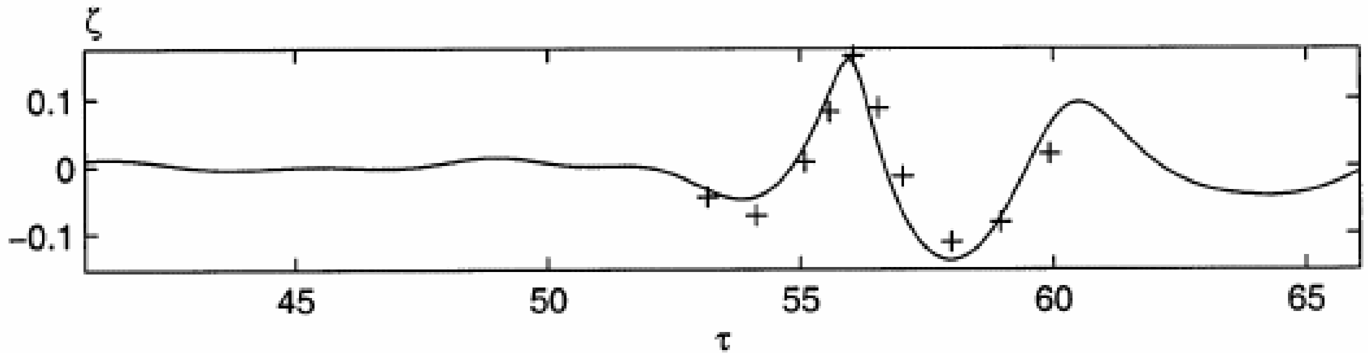
Using structured mesh

- Comparison with experiment
- 2-D floating bodies in forced motions ;
- 2-D resonance problems (second order & fully nonlinear);
- 2-D solitary wave problems;
- 3-D sloshing problems .

Using structured mesh…

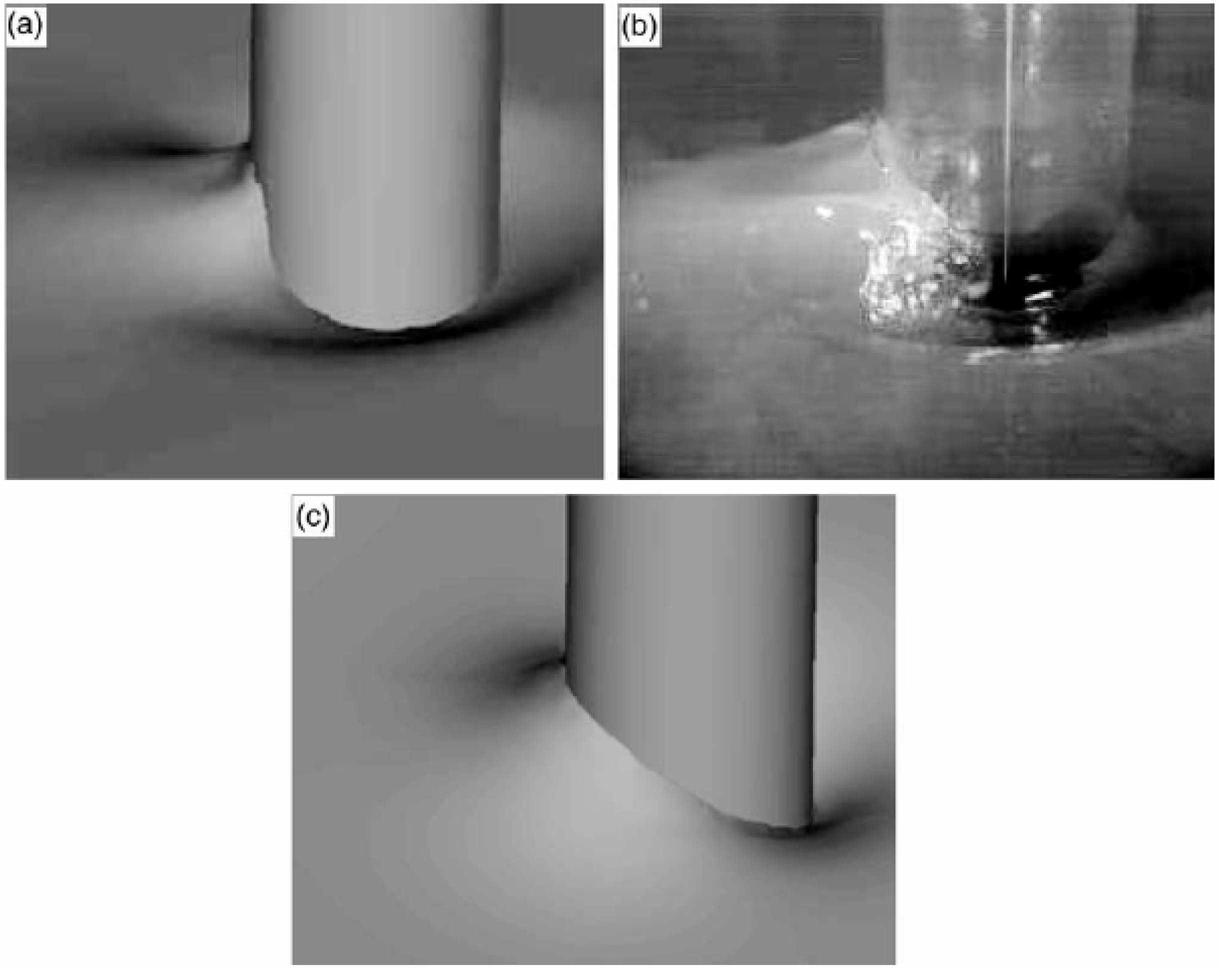


The history of irregular wave at x=3.436 for α =0.612 (Solid line: shorter tank L=14.64; Dash line: longer tank L=44.64; +: experimental data from Nestegard, 1999)



The history of irregular wave at x=3.436 for α =0.749 (Solid line: shorter tank L=14.64; Dash line: longer tank L=44.64; +: experimental data from Nestegard, 1999)

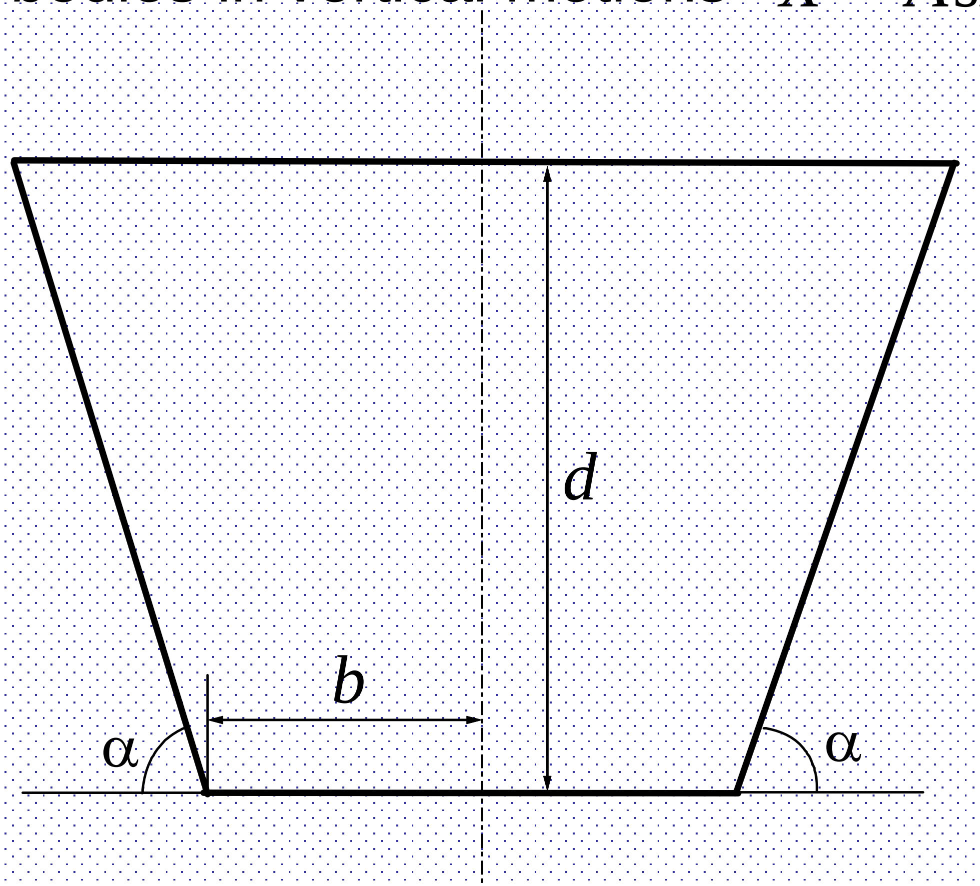
Using structured mesh....



Wave profile at time 0.4s (a) Fully nonlinear result; (b) Experimental result (Retzler et al, 2000); (c) Linear result

Using structured mesh

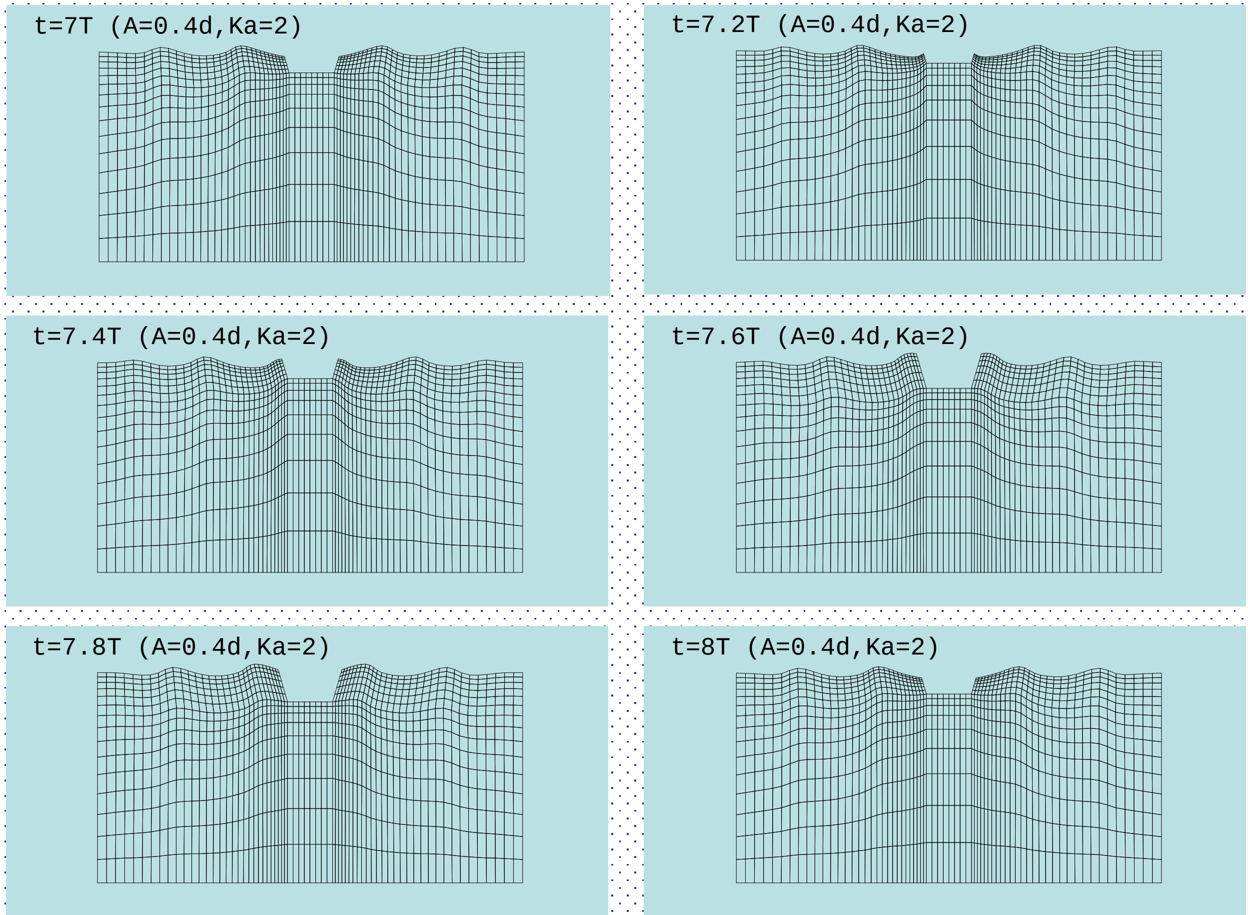
2-D floating bodies in vertical motions $X = A\sin \omega t$



Dimension of trapezoid-shape body

Using structured mesh…

- 2-D floating bodies in forced motions $X = A\sin\omega t$



snapshots of meshes for single cylinder at different time steps (α=75°)

Using structured mesh…

- 2-D floating bodies in forced motions $X = A\sin\omega t$

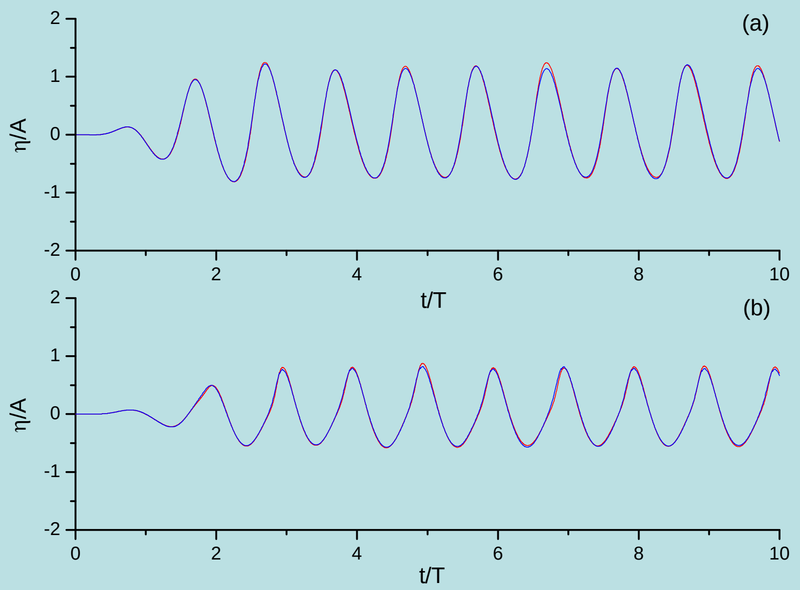
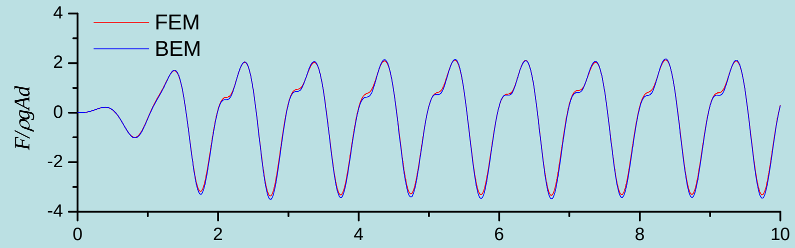


Fig. Wave histories at (a) x=-2d and (b) x=-3d (Ka=2.0 and A/d=0.4)
FEM; BEM

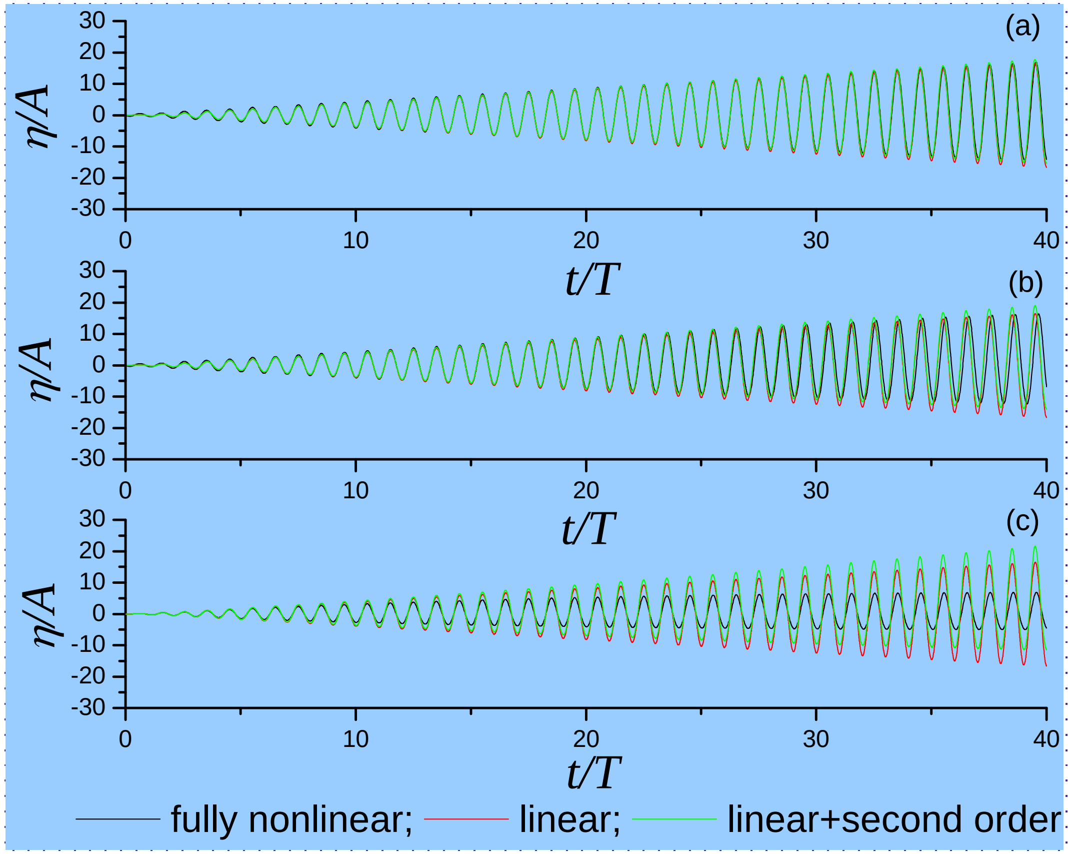


Comparison of vertical forces between the FEM and the BEM at *Ka*=2.0 and *A*=0.4*d*

Comparisons of waves and forces between FEM and BEM

Using structured mesh…

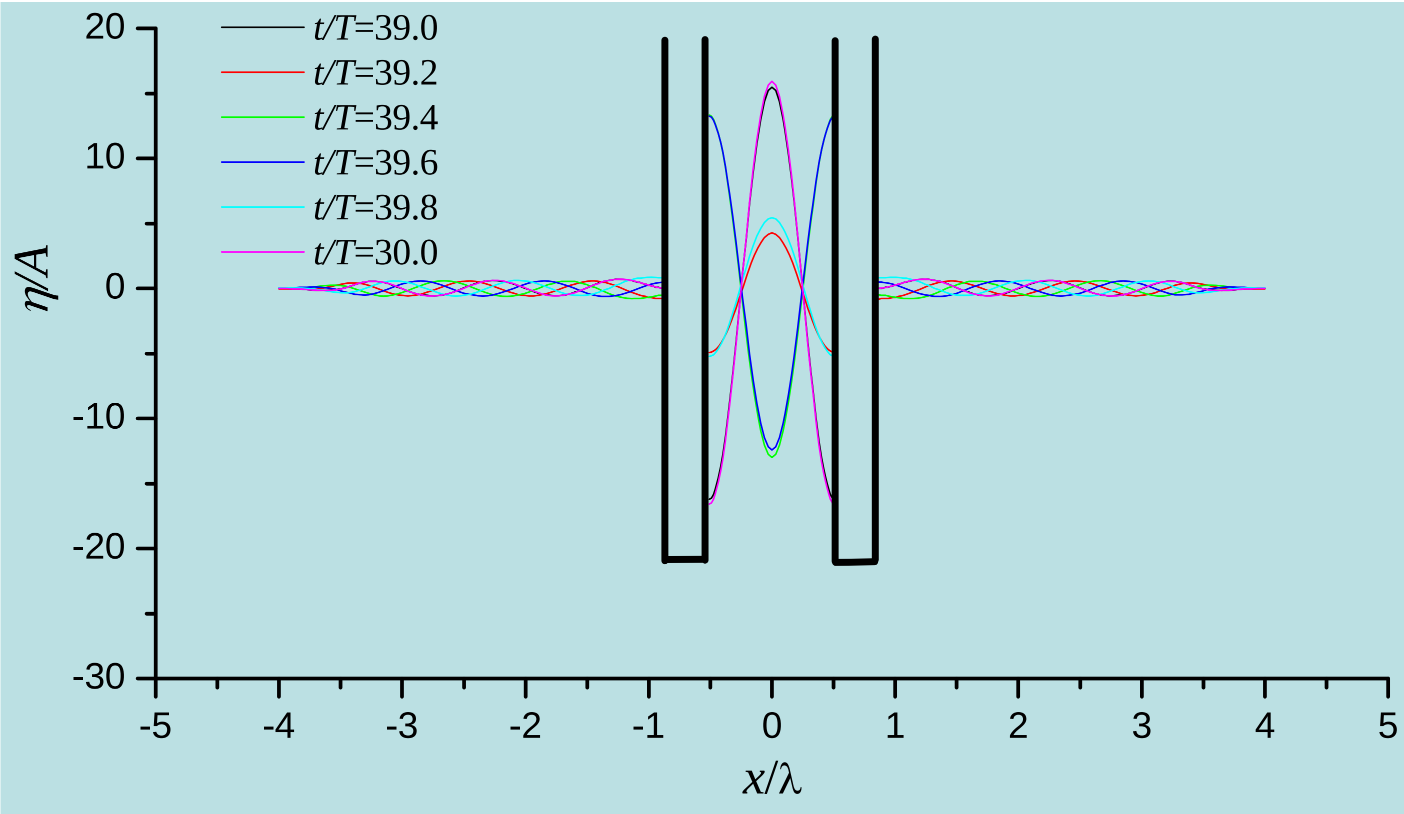
- 2-D second-order analysis in time domain: two rectangular cylinders in heave $X = A\sin\omega t$ (Wang & Wu, 2008, Ocean Eng.)



Comparison of waves at the right side of cylinder one at the first order resonant frequency (a) A=0.0125d (b) A=0.025d (c) A=0.05d

Using structured mesh…

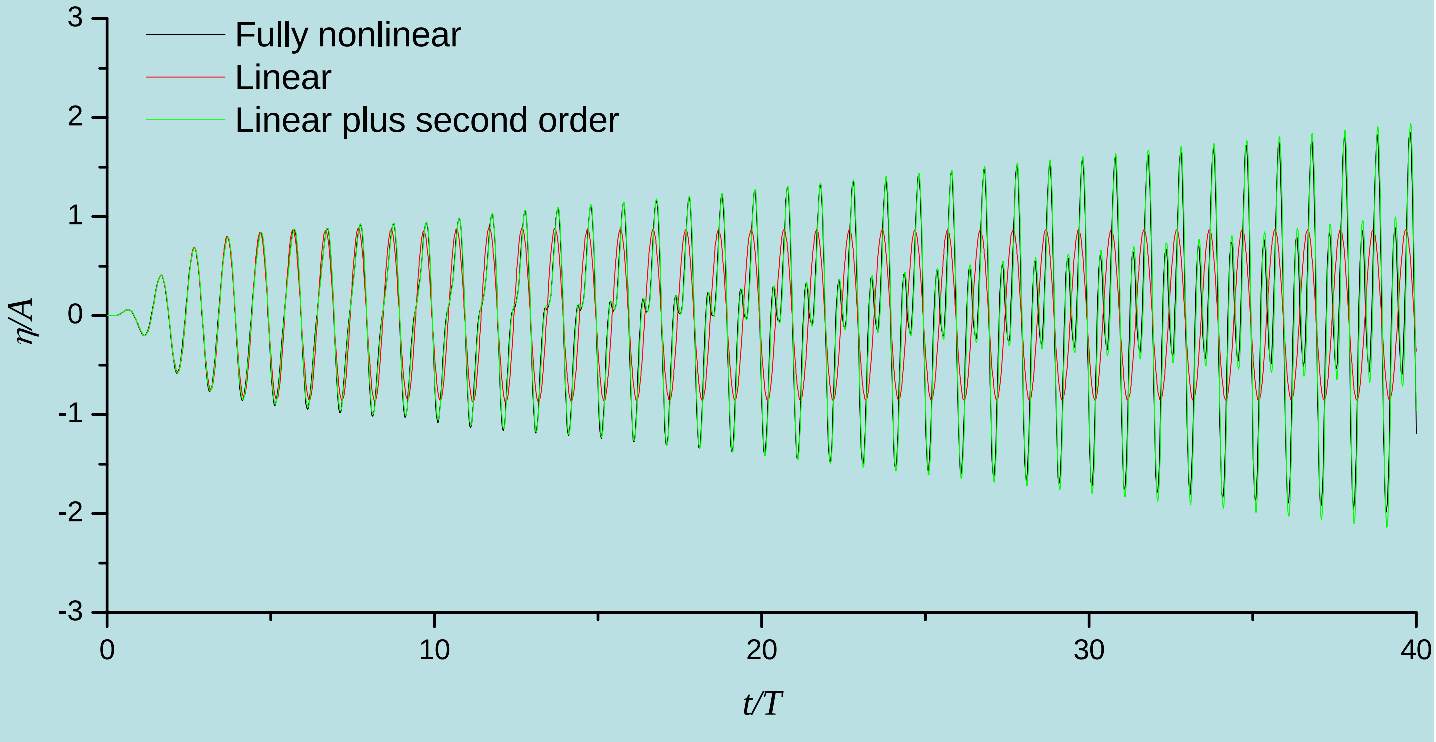
- 2-D second-order analysis in time domain: two rectangular cylinders in heave $X = A\sin\omega t$ (Wang & Wu, 2008, Ocean Eng.)



Linear free surface profiles at the first order resonant frequency

Using structured mesh…

- 2-D second-order analysis in time domain: two rectangular cylinders in heave $X = A\sin\omega t$ (Wang & Wu, 2008, Ocean Eng.)

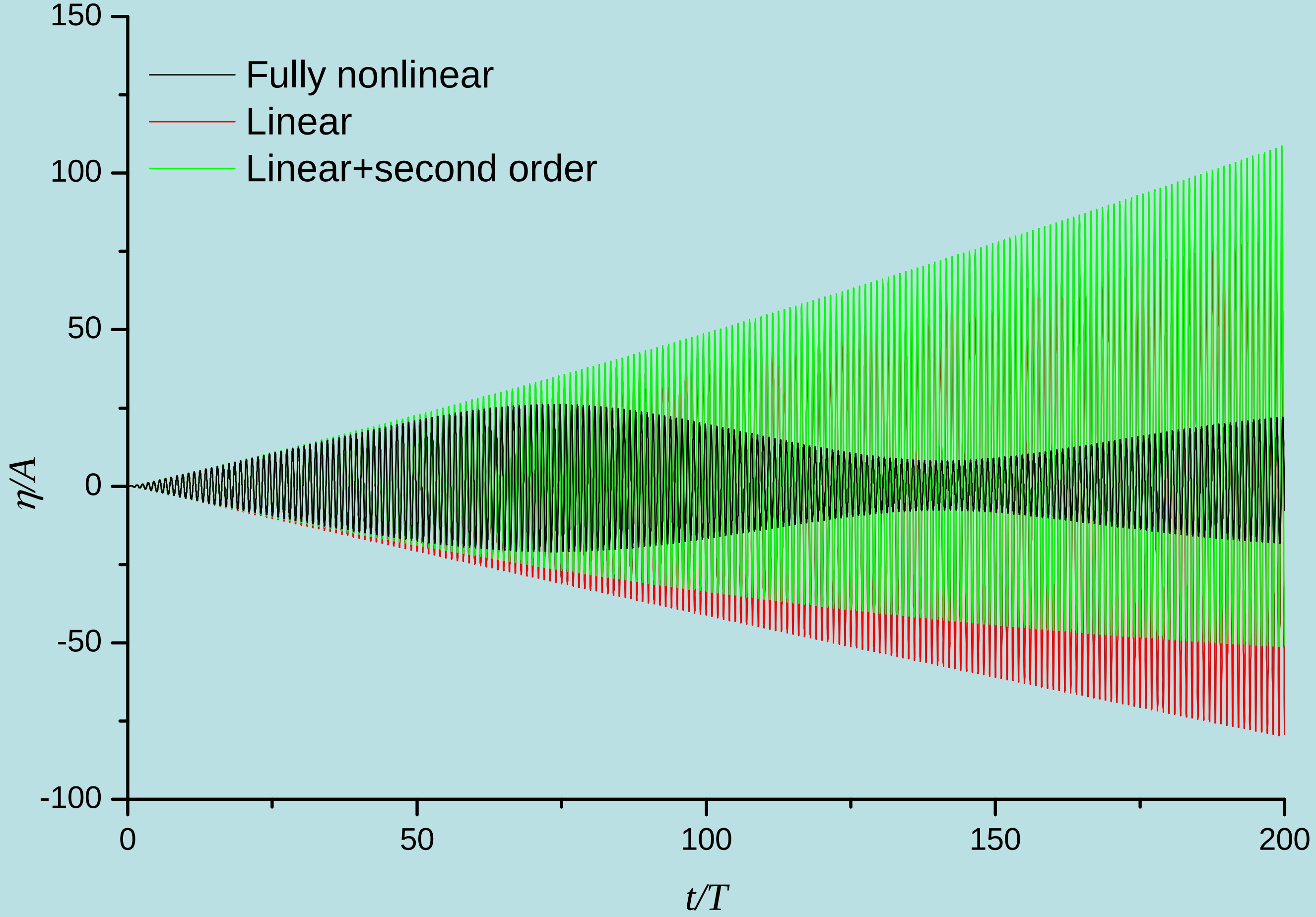


Comparison of waves at the right side of cylinder one $\omega=0.5\omega_4'$ and $A=0.05d$

(at the second order resonant frequency)

Using structured mesh…

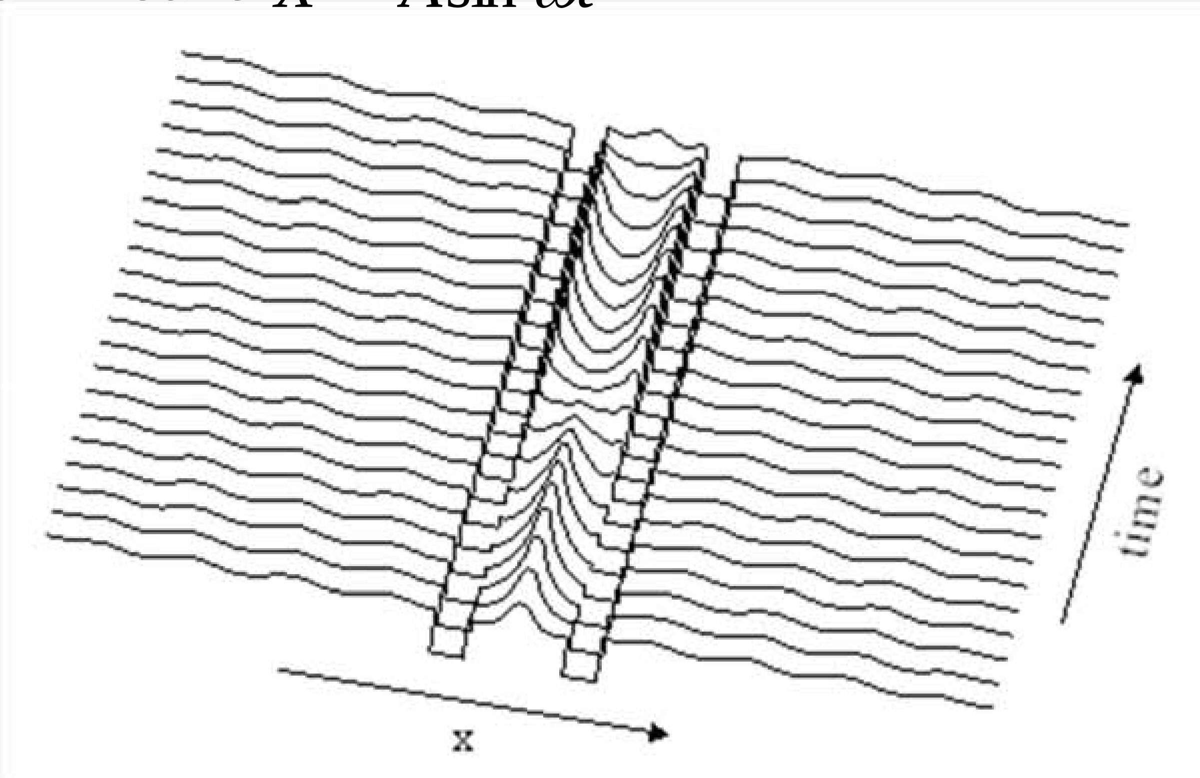
- 2-D fully nonlinear analysis : two rectangular cylinders in heave $X = A\sin\omega t$ (A=0.0125d)



Comparion of waves at the right side of cylinder one at the first order resonant frequency

Using structured mesh…

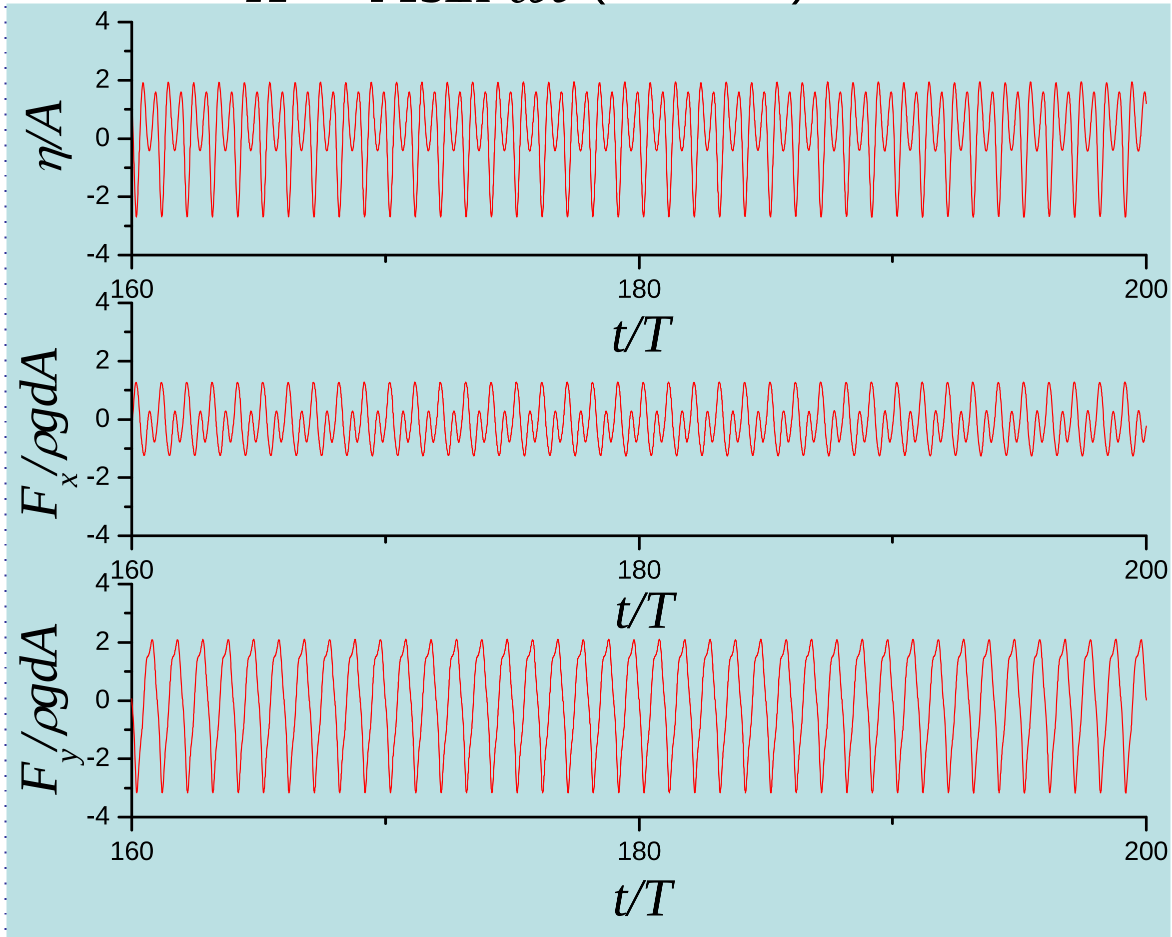
- 2-D fully nonlinear analysis : two rectangular cylinders in heave $X = A\sin\omega t$



Wave profiles at A=0.05d (at the first order resonant frequency)

Using structured mesh…

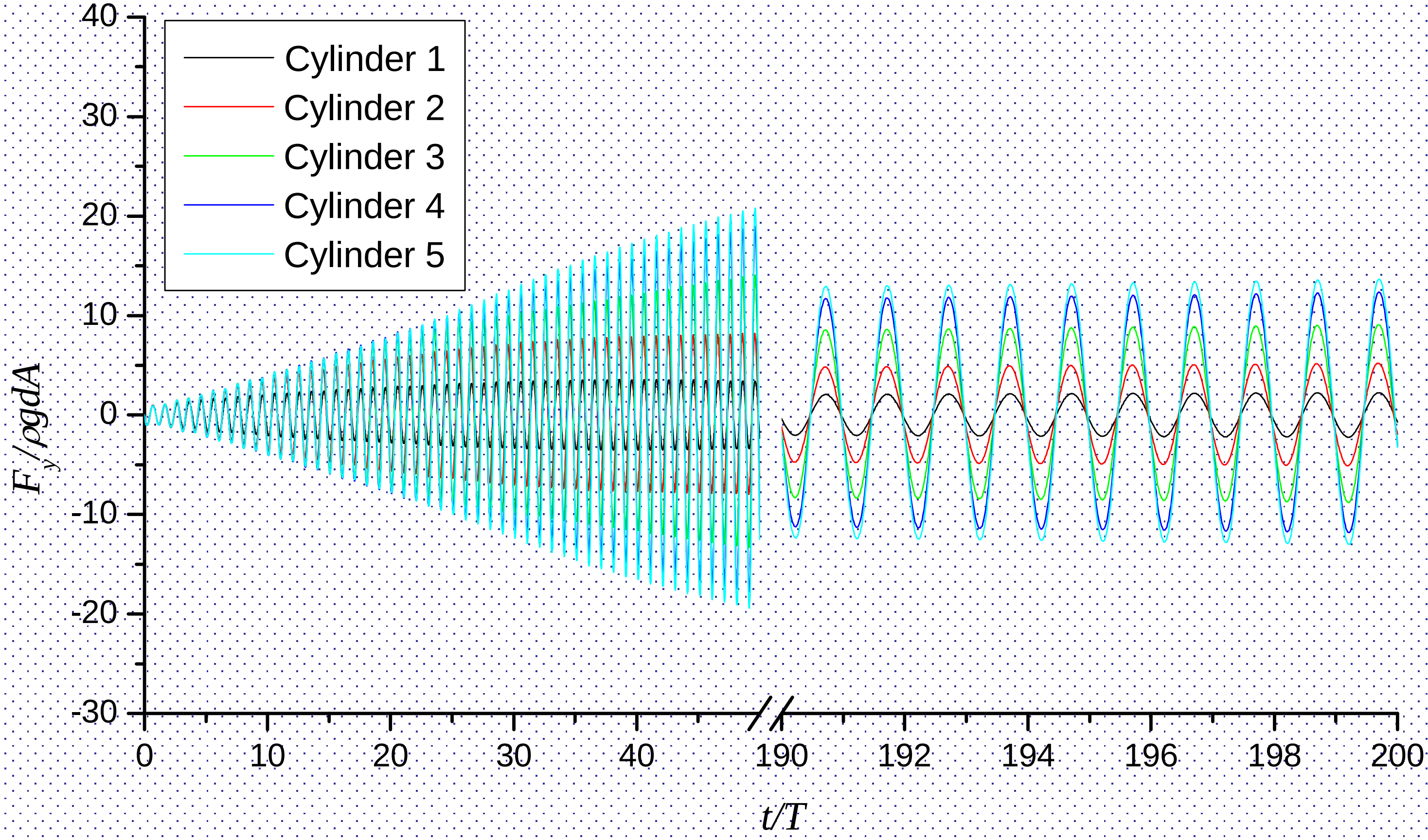
- 2-D fully nonlinear analysis : two rectangular cylinders in heave $X = A\sin\omega t$ ($A=0.2d$)



Waves and forces at the second order resonant frequency

Using structured mesh…

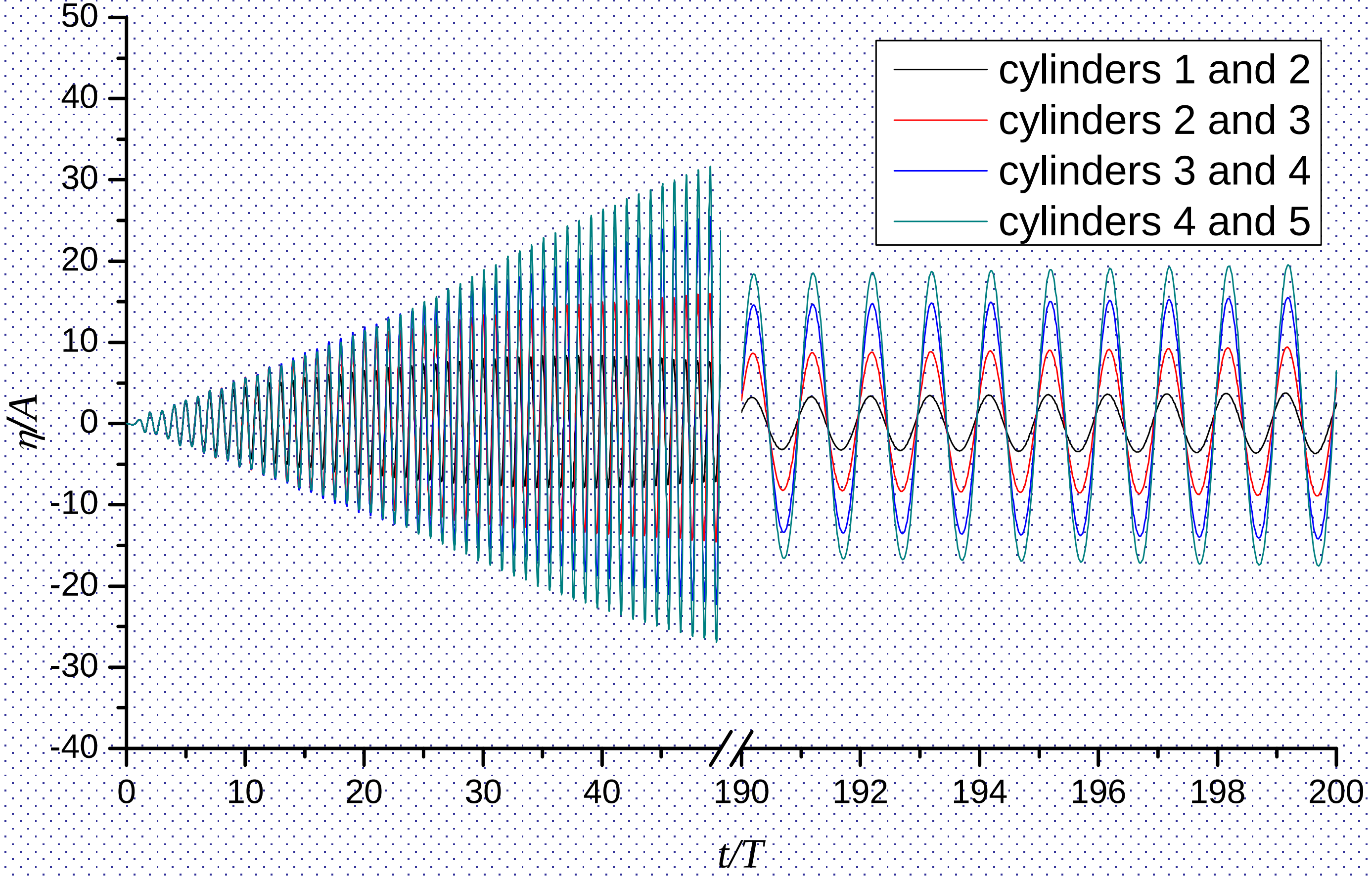
- 2-D fully nonlinear analysis : nine wedge-shaped cylinders in heave $X = A\sin\omega t$



Histories of hydrodynamic forces at first order resnonat frequency and A=0.005d

Using structured mesh…

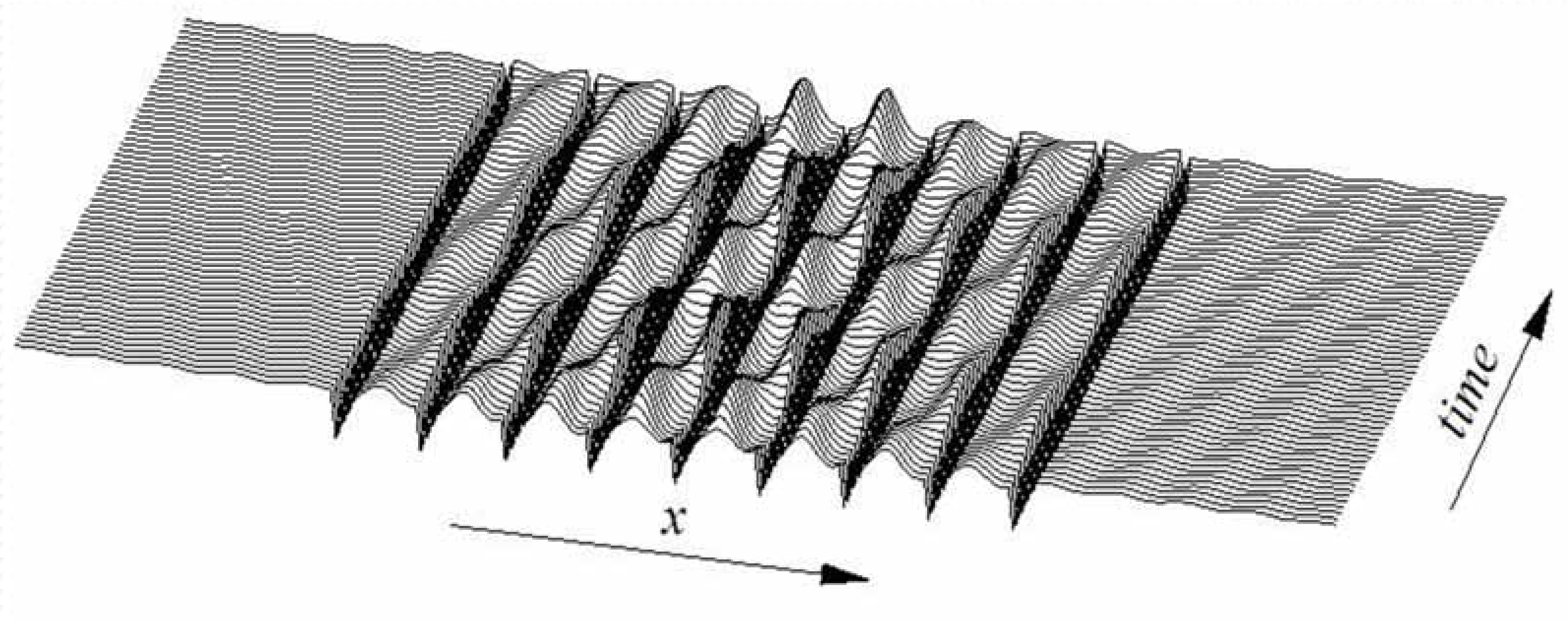
- 2-D fully nonlinear analysis : nine wedge-shaped cylinders in heave $X = A\sin\omega t$



Histories of waves at the midpoint of two neighboring cylinders at first order resnonat frequency and A=0.005d

Using structured mesh…

- 2-D fully nonlinear analysis : nine wedge-shaped cylinders in heave $X = A\sin\omega t$

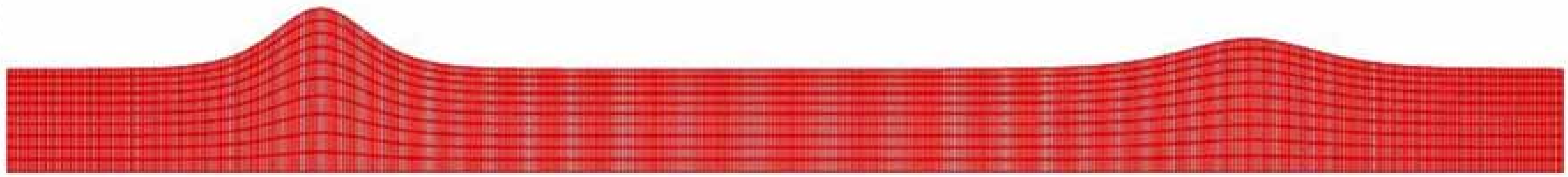


Wave profiles at t/T=45, 45.04, 45.08,…,47.32 at A=0.04d.

(at the first order resonant frequency)

Using structured mesh…

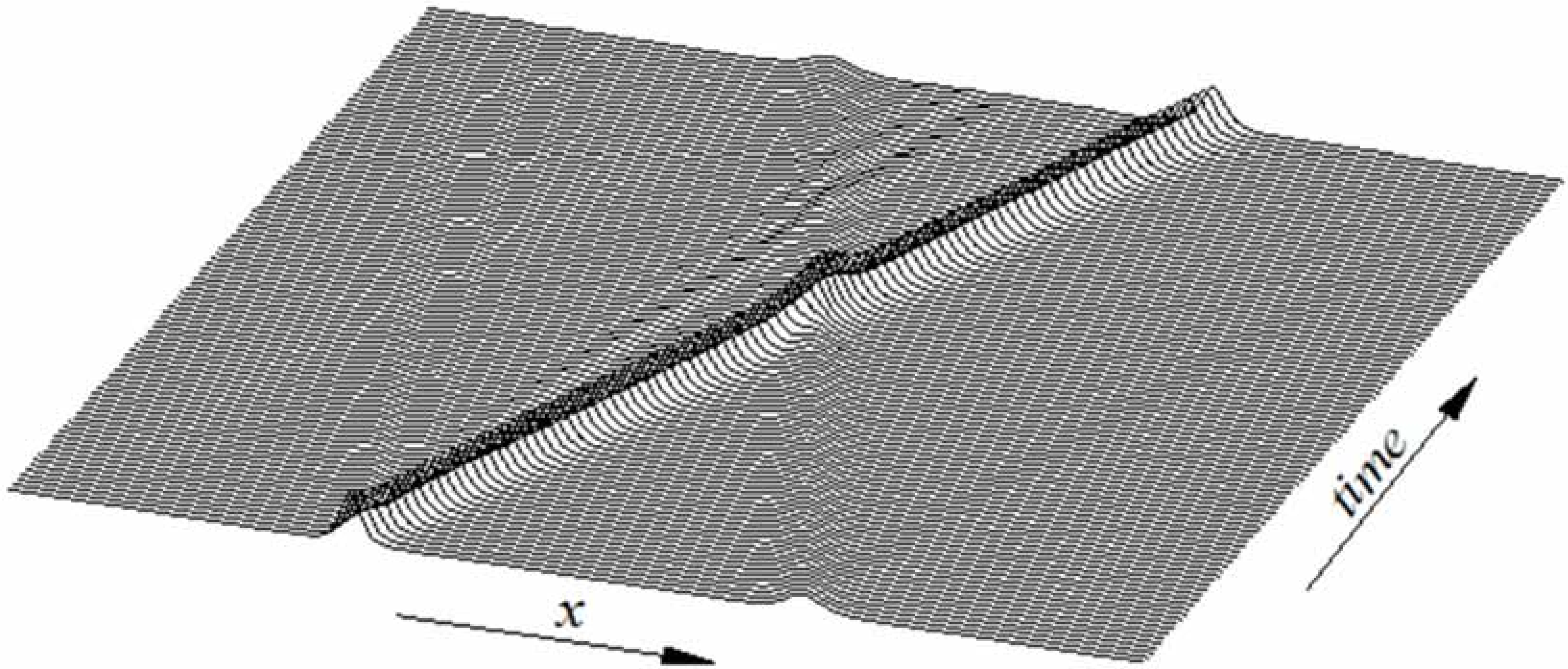
- 2-D solitary wave problems



Finite element mesh for two solitary waves colliding with each other (8-node quadrilateral element)

Using structured mesh…

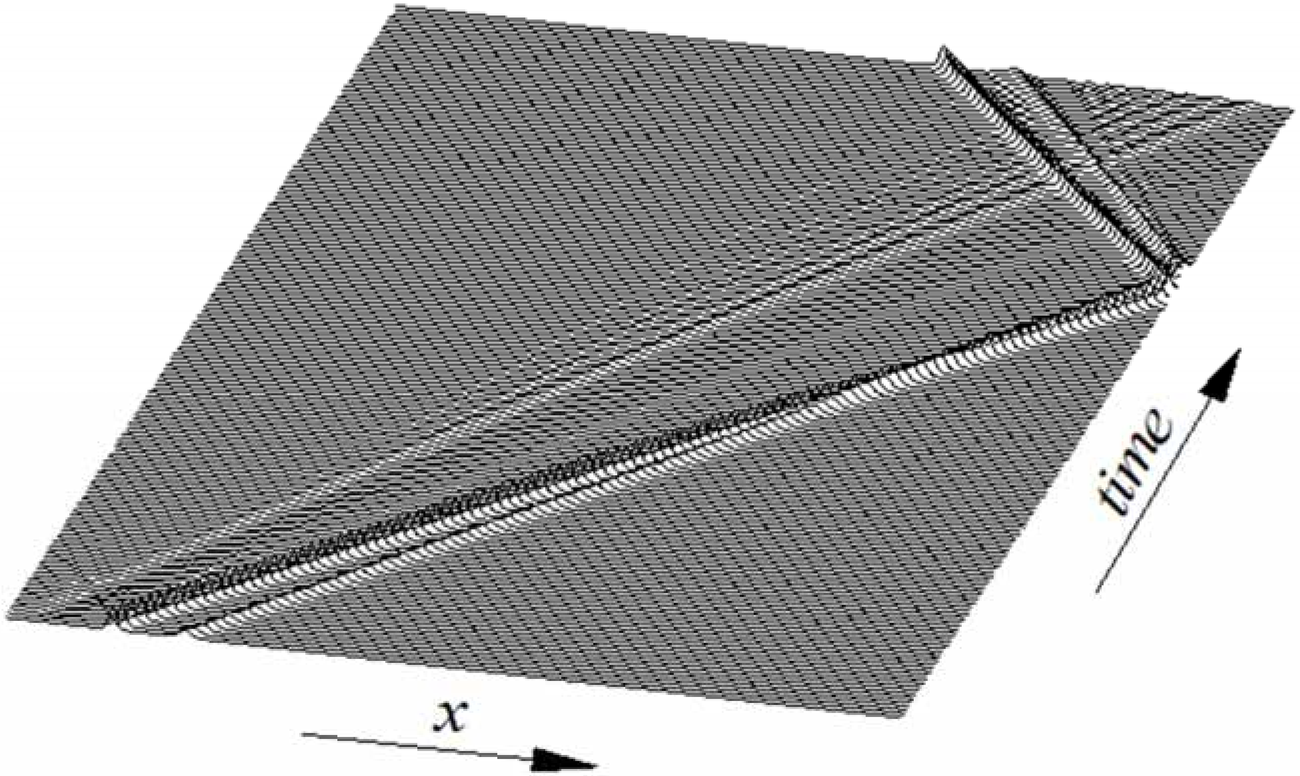
- 2-D solitary wave problems



Wave profiles for two waves colliding with each other ($L=10L_{eff}$, $H_1=0.6h$, $x_{p1}=3L_{eff}$, $H_2=0.2h$, $x_{p2}=7L_{eff}$, where Leff is the efficient wave length)

Using structured mesh…

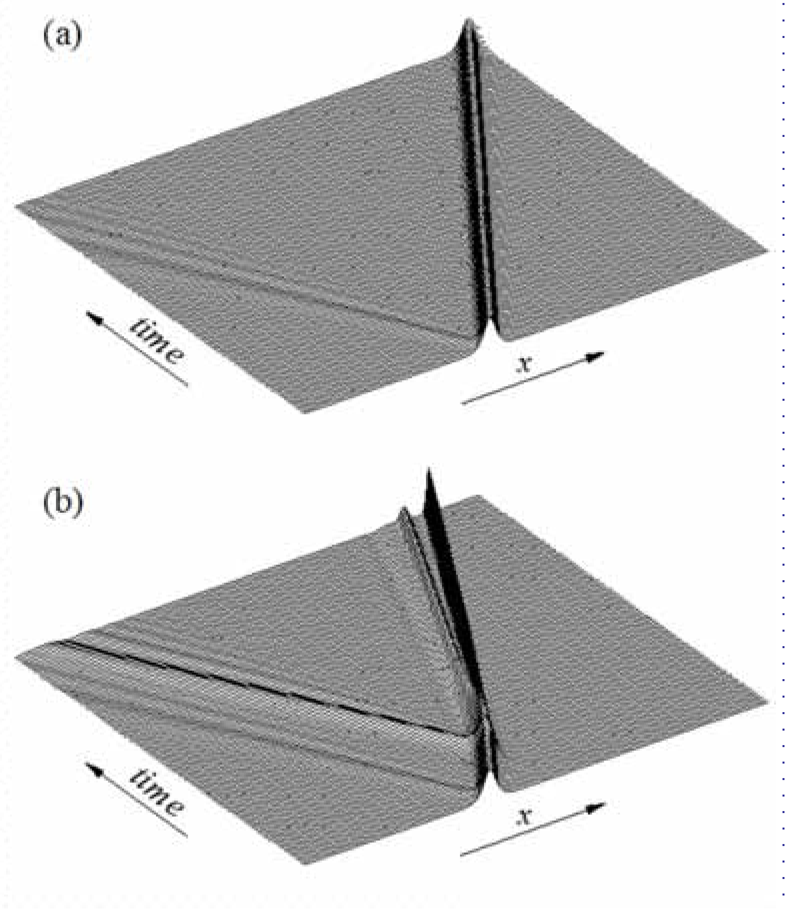
- 2-D solitary wave problems



Wave profiles for one wave overtaking another
(*L*=25*Leff*, *H*1=0.6*h*, *xp*1=3*Leff*, *H*2=0.2*h* *xp*2=5*Leff*)

Using structured mesh…

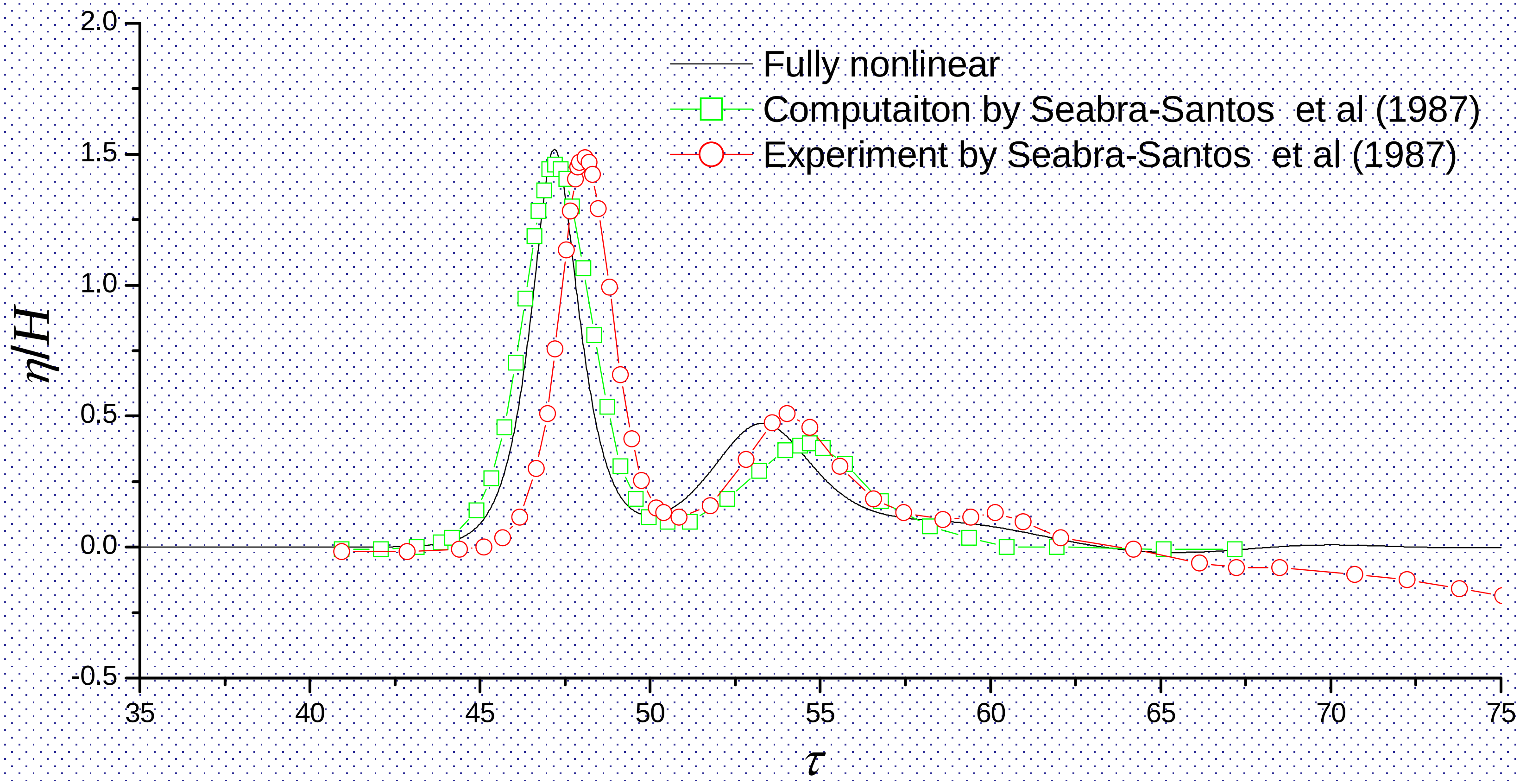
- 2-D solitary wave problems



Wave profiles for a solitary wave propagates over a step
(a) without step (b) with step

Using structured mesh…

- 2-D solitary wave problems

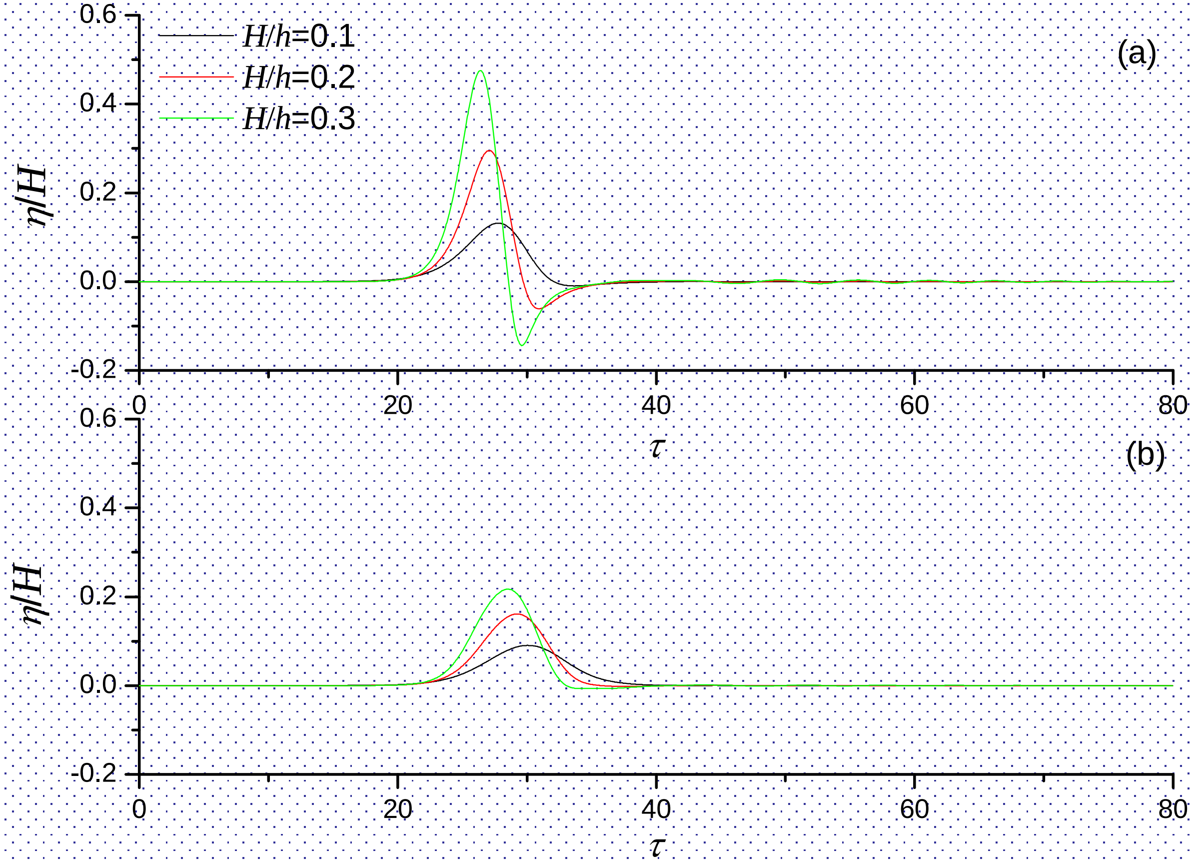


Comparison of histories of waves at x=21m

(a solitary wave propagates over a step)

Using structured mesh…

- 2-D solitary wave problems

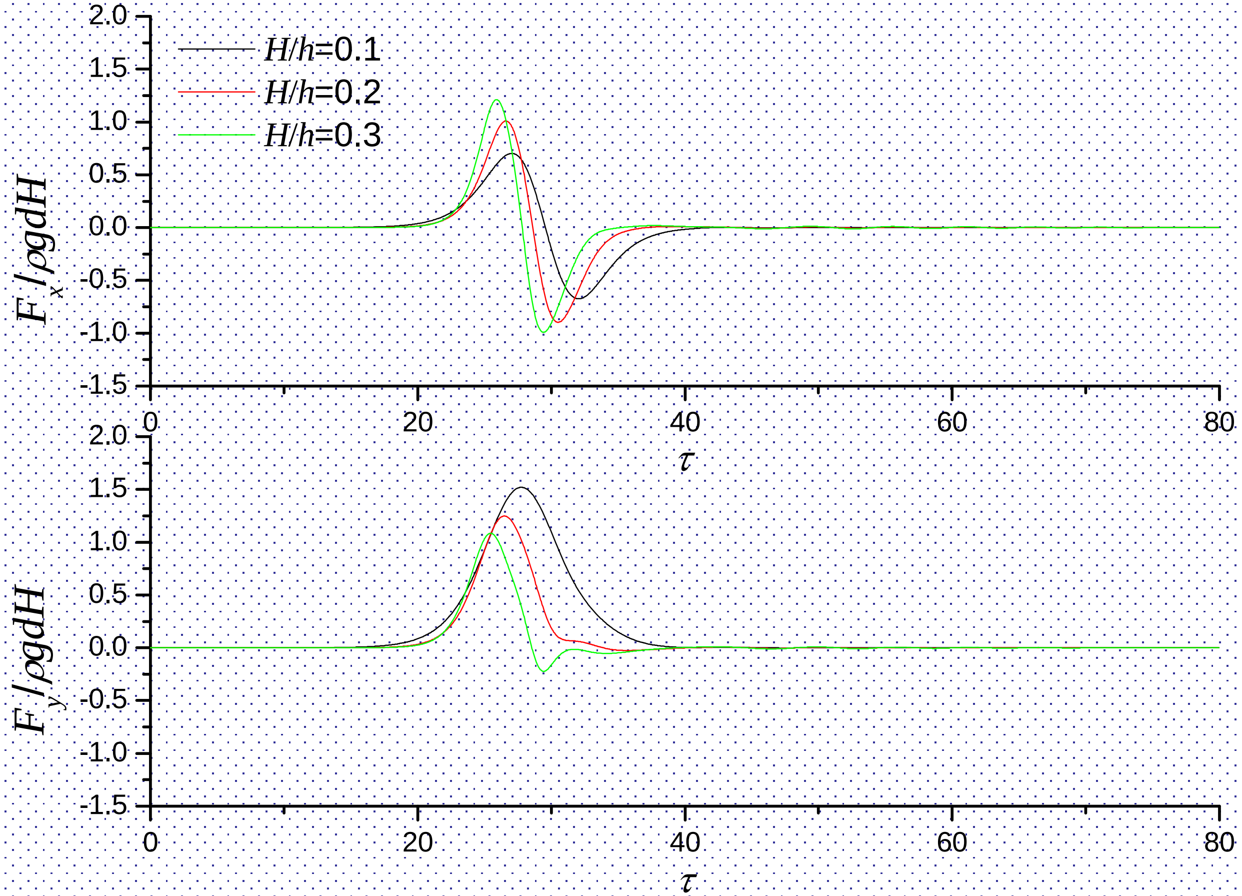


Histories of waves at the (a) left side and (b) right side of the cylinder

(Interactions of solitary waves with a floating rectangular cylinder)

Using structured mesh…

- 2-D solitary wave problems

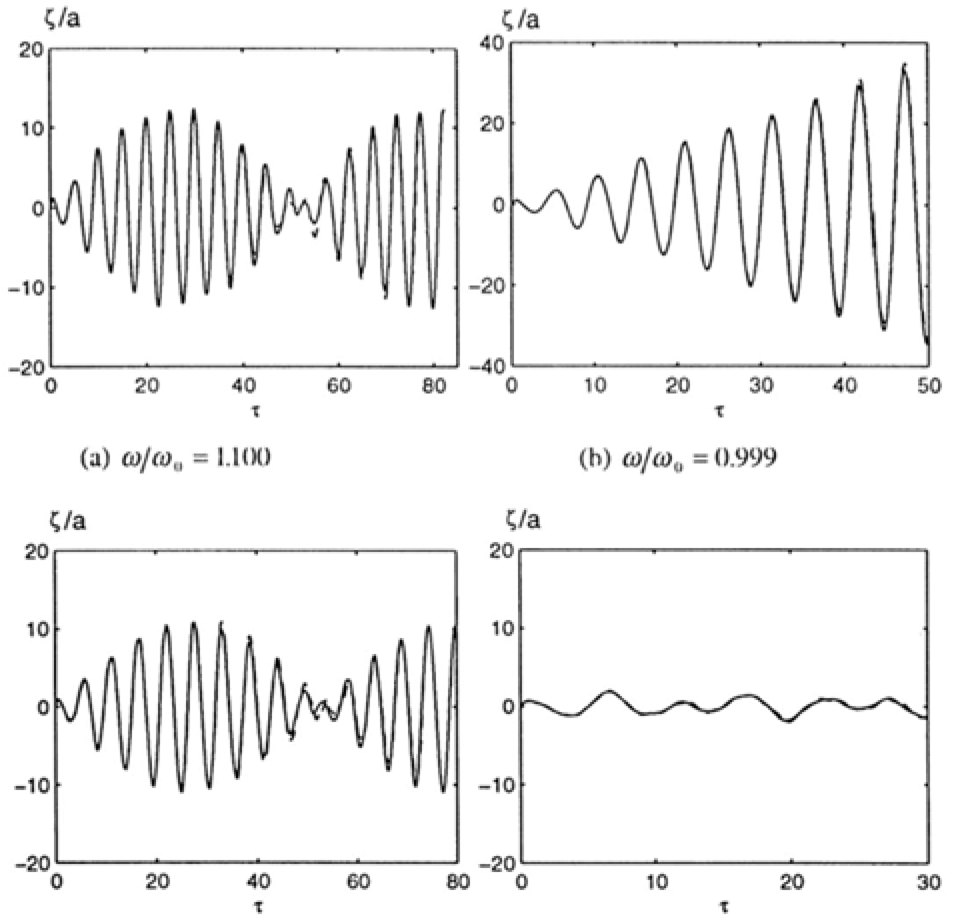


Histories of hydrodynamic forces on the cylinder

(Interactions of solitary waves with a floating rectangular cylinder)

Using structured mesh…

- 3-D sloshing problems (Wu, Ma Eatock Taylor 1998, Applied Ocean Res)

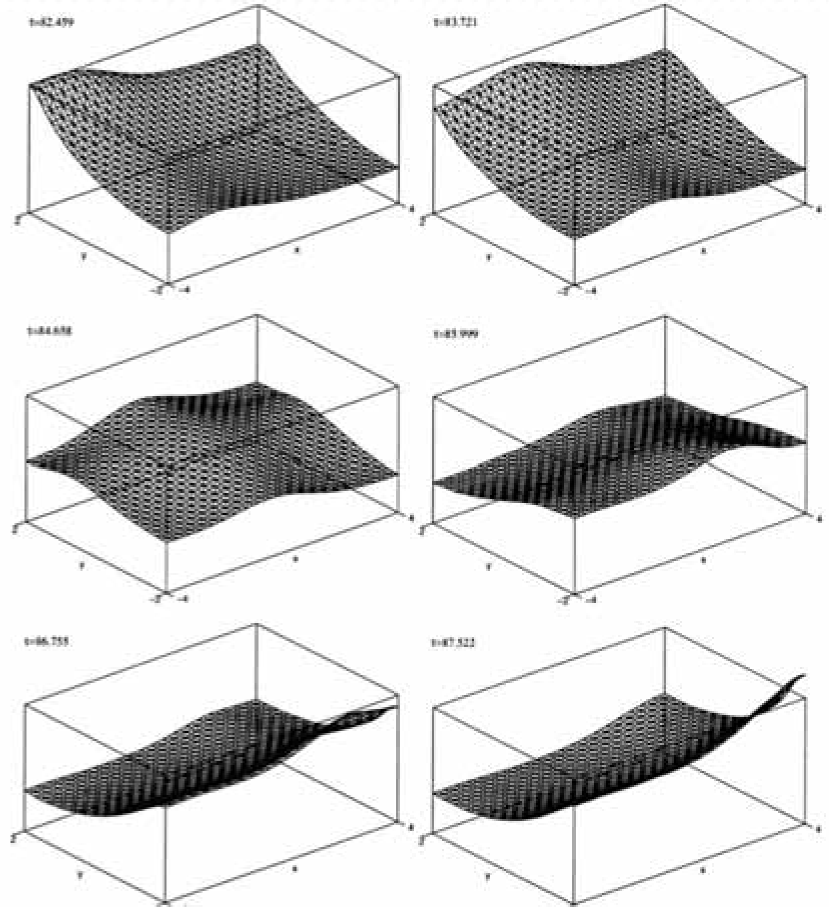


(a) $\omega/\omega_0 = 1.100$ (b) $\omega/\omega_0 = 0.999$

(c) $\omega/\omega_0 = 0.900$ (d) $\omega/\omega_0 = 0.583$

Using structured mesh…

- 3-D sloshing problems (Wu, Ma Eatock Taylor 1998, Applied Ocean Res)



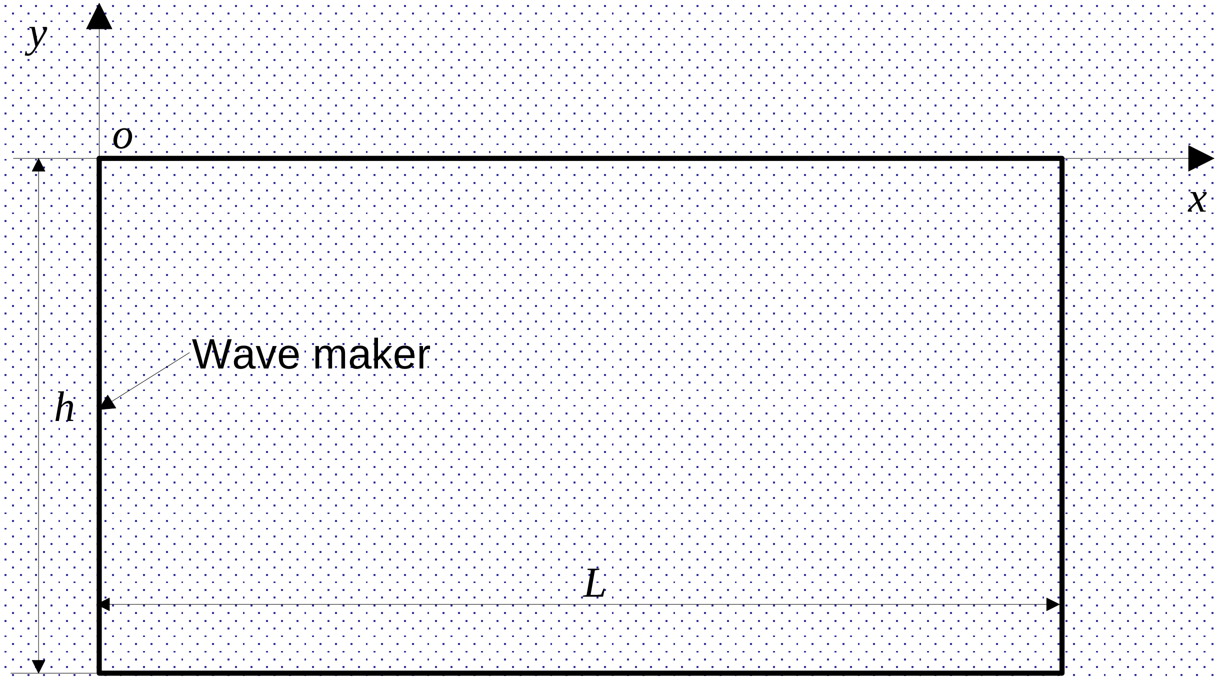
Snapshots of free surface in some cases

Using unstructured mesh

- 2-D wave-making problem;
- 2-D wave radiation by floating wedged-shape bodies;
- 2-D solit
- 3-D large amplitude motions of vertical cylinders and motions of a floating FPSO;
- 3-D wave-making problem for non-wall-sided cylinders;
- 3-D second-order diffraction by multiple cylinders in the time domain;
- 3-D wave-making problem for multiple cylinders;
- 3-D wave radiation by multiple cylinders.;

Using unstructured mesh…

- 2-D wave-making problems (Wang & Wu 2006, J. Fluids & Strucs.)

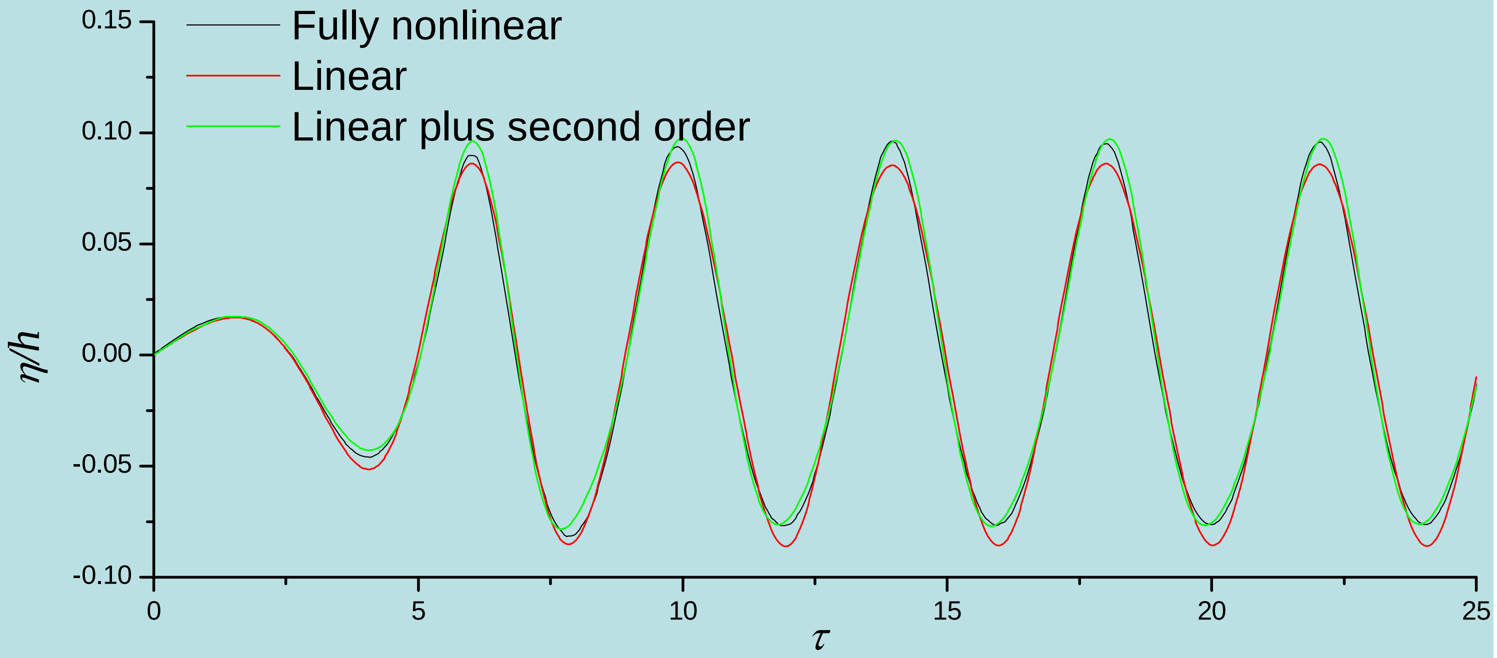


Sketch of a tank

$$(X = -A\sin\omega t, \omega = 1.5539\sqrt{g/h}, \tau = t/\sqrt{h/g})$$

Using unstructured mesh…

- 2-D wave-making problems (Wang & Wu, 2006)

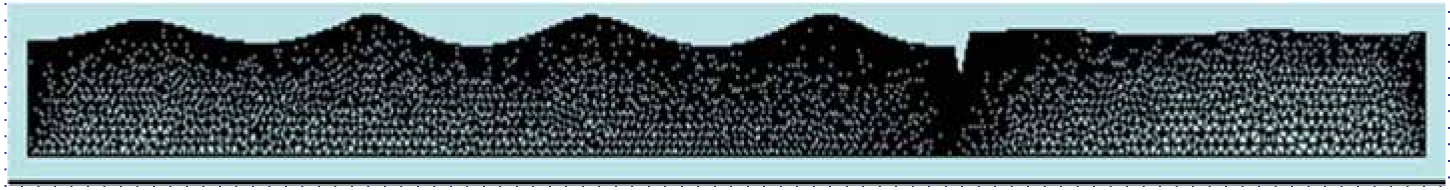


Comparison of wave histories at x=1.167h (A/h=0.05)

Wave profile in the tank at τ=24.26 (A/h=0.1)

Using unstructured mesh…

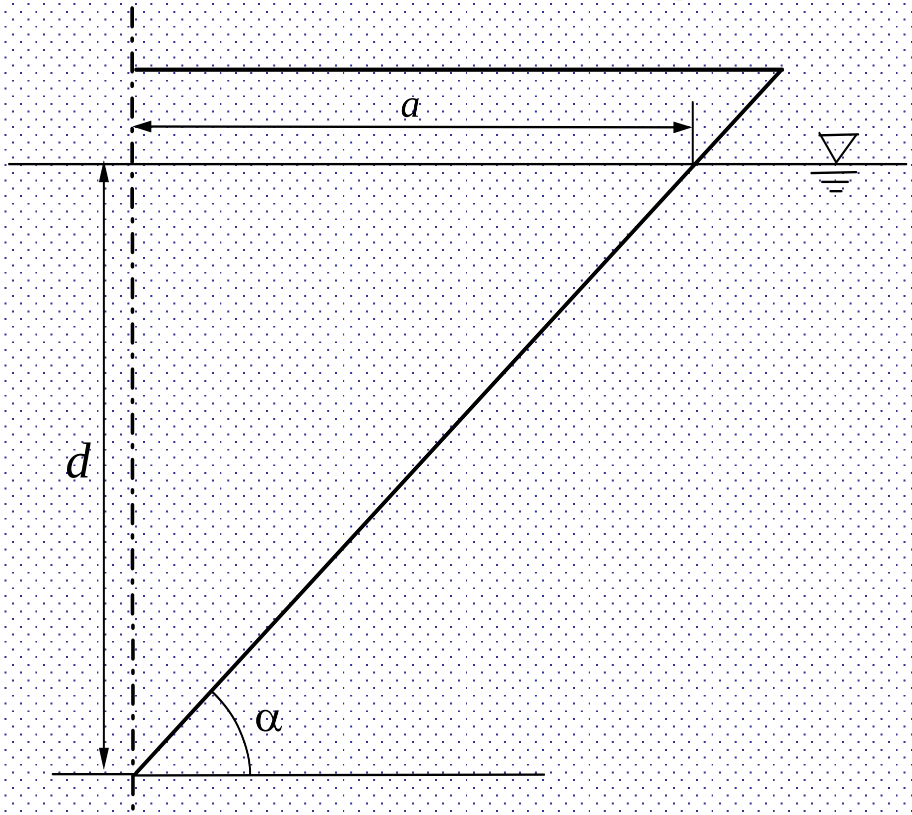
- 2-D wave-making problems (Wang & Wu, J. Fluids & Struc. 2006)



Wave profile at τ=58.63 (a wedged-shape body in the tank)

Using unstructured mesh…

- 2-D wave radiation by floating wedged-shape bodies (Wang & Wu 2006, J. Fluids & Strus.)

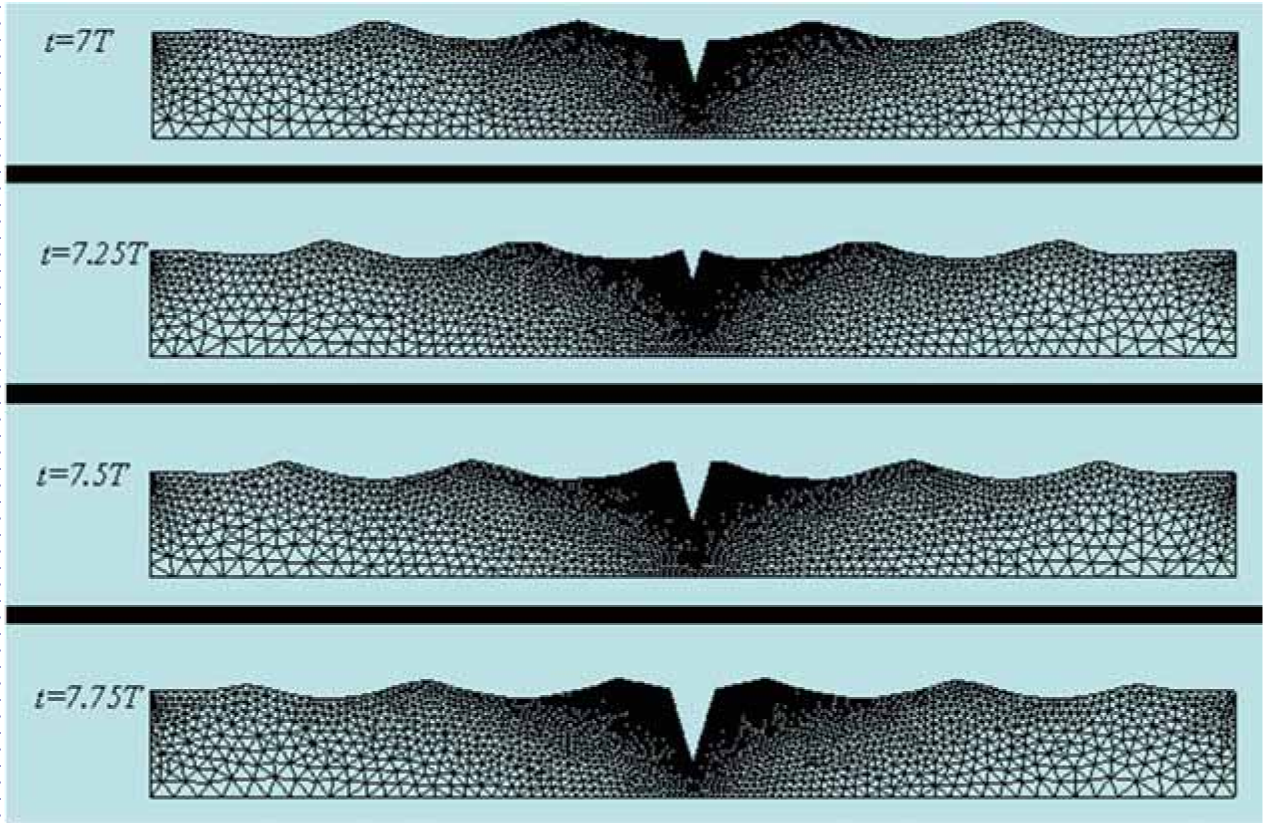


Sketch of a floating wedge

$$(h = 2d, Y = A\sin\omega t, \overline{\omega} = \omega\sqrt{g/h})$$

Using unstructured mesh…

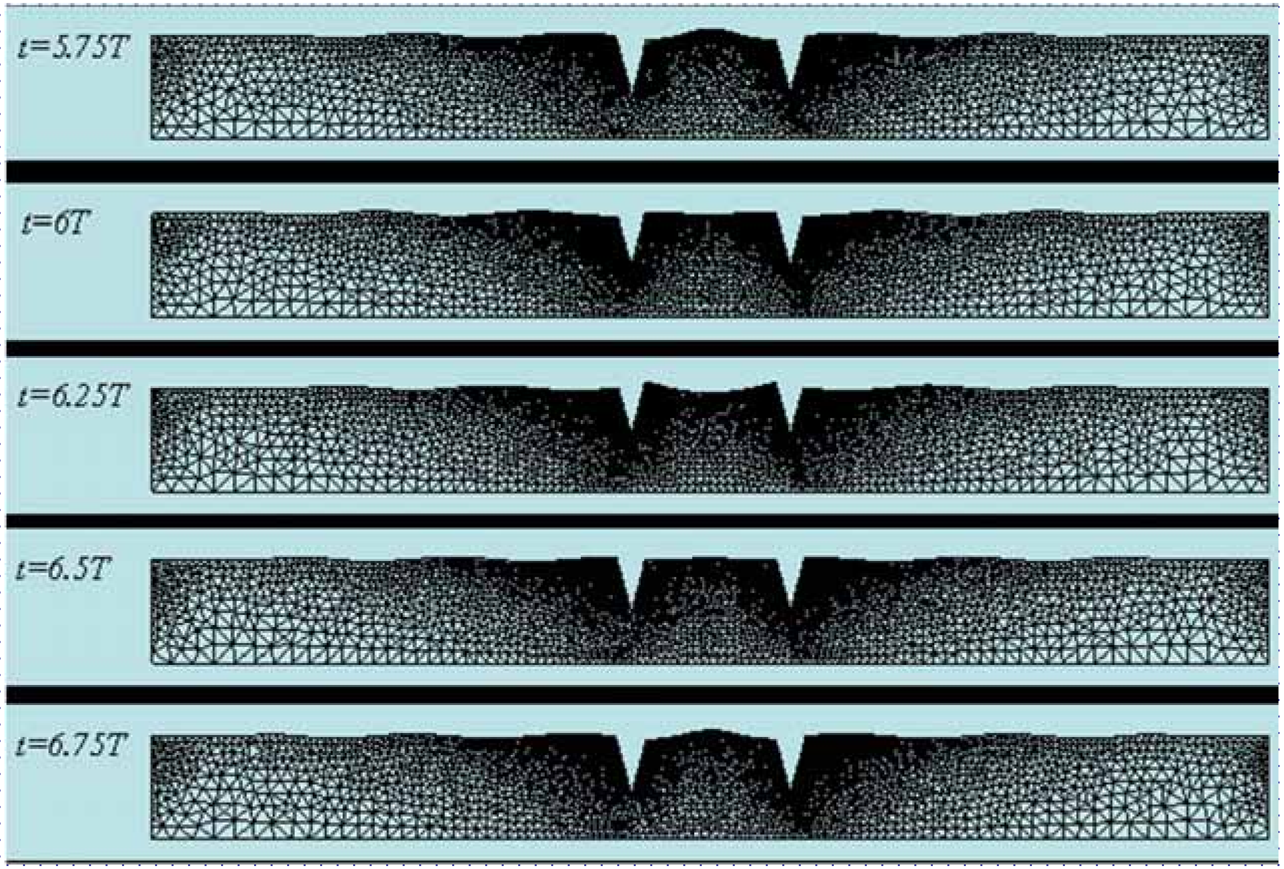
- 2-D wave radiation by floating wedged-shape bodies (Wang & Wu 2006, J. Fluids & Strucs.)



Single wedge $(\overline{\omega}=2, A=0.4d, \alpha=75^o)$

Using unstructured mesh…

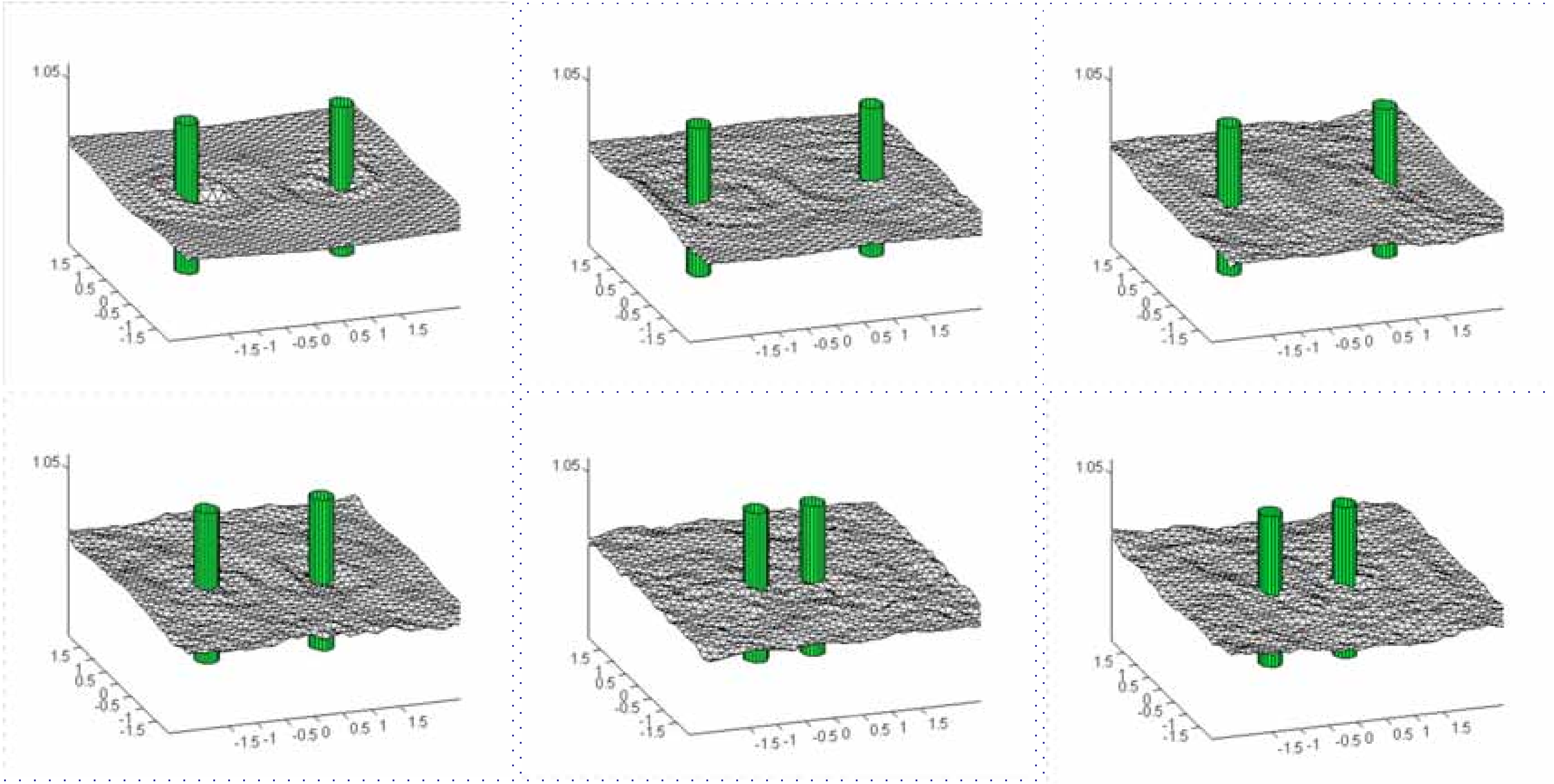
- 2-D wave radiation by floating wedged-shape bodies (Wang & Wu 2006, J. Fluids & Strucs.)



Twin wedges $(\overline{\omega} = 2, A = 0.1d, \alpha = 75^o)$

Using unstructured mesh…

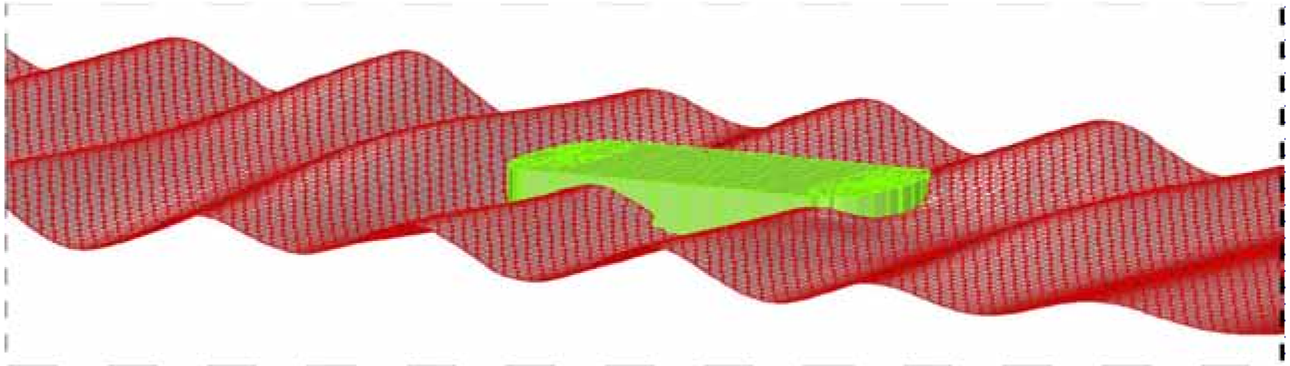
- 3-D large motions of vertical cylinders (Wu & Hu, 2004, Proc. Roy. Soc. London)



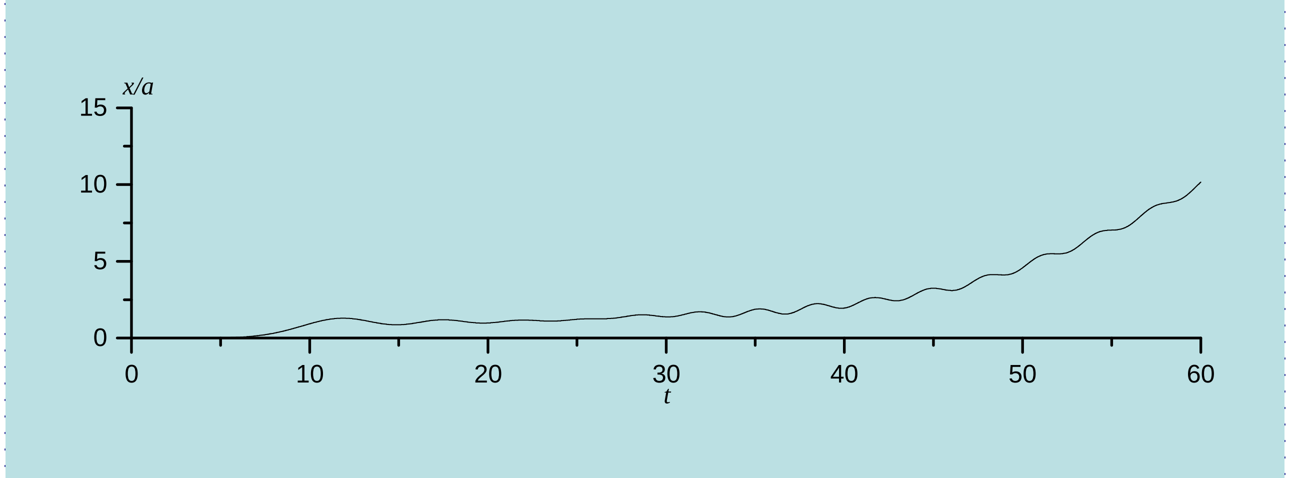
Wave profiles by two cylinders undergoing periodic oscillation at $t=T, 2T, \ldots, 6T$

Using unstructured mesh…

- Motions of a FPSO in a tank (Wu & Hu 2004, Proc. Roy. Soc. London)



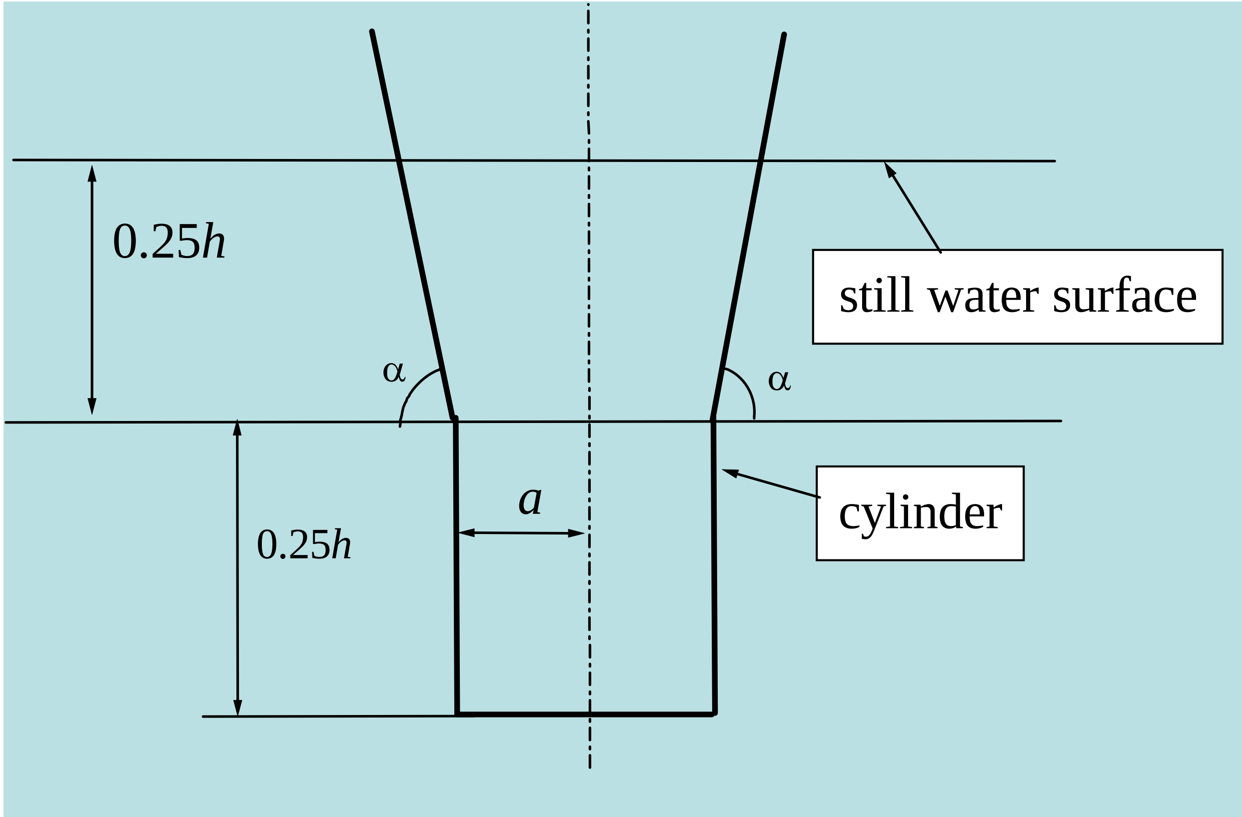
Wave profile around a FPSO



Time history of the displacement of the FPSO at $A = 0.01h$

Using unstructured mesh…

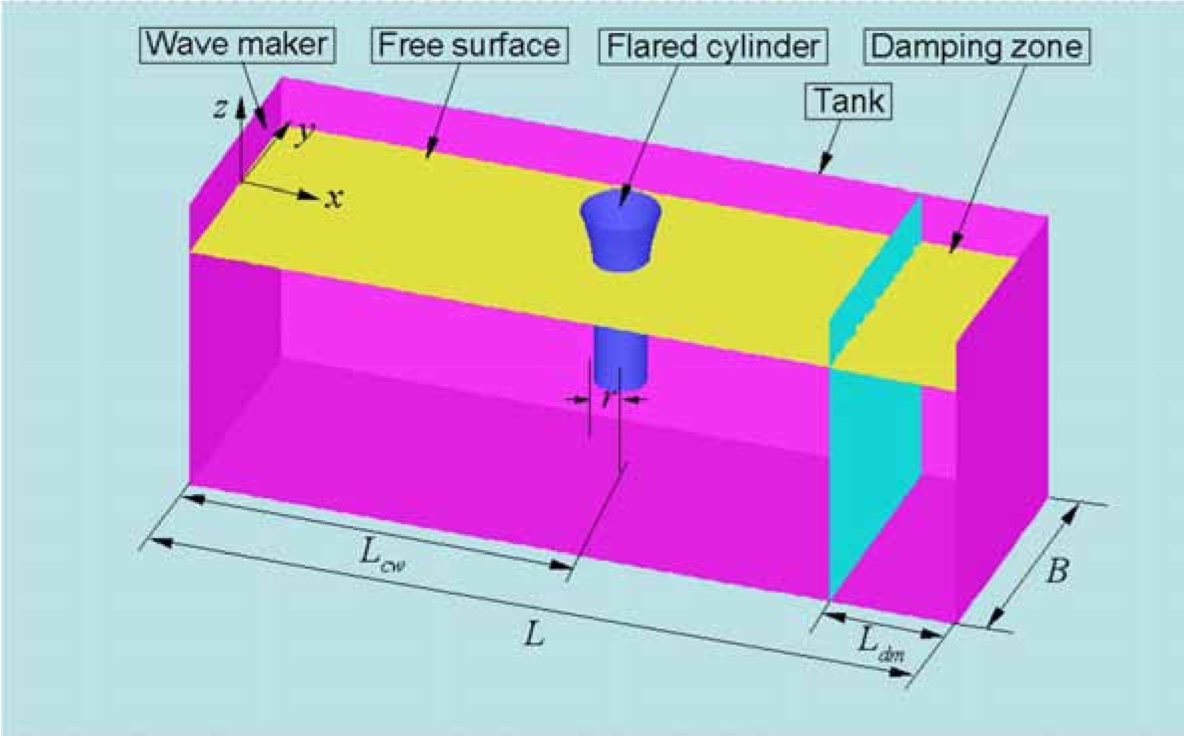
- 3-D wave-making problem for non-wall-sided cylinders (Wang, Wu & Drake, Ocean Eng. 2007)



Dimension of truncated cylinder with flare

Using unstructured mesh…

- 3-D wave-making problem for non-wall-sided cylinders (Wang, Wu & Drake, Ocean Eng. 2007)

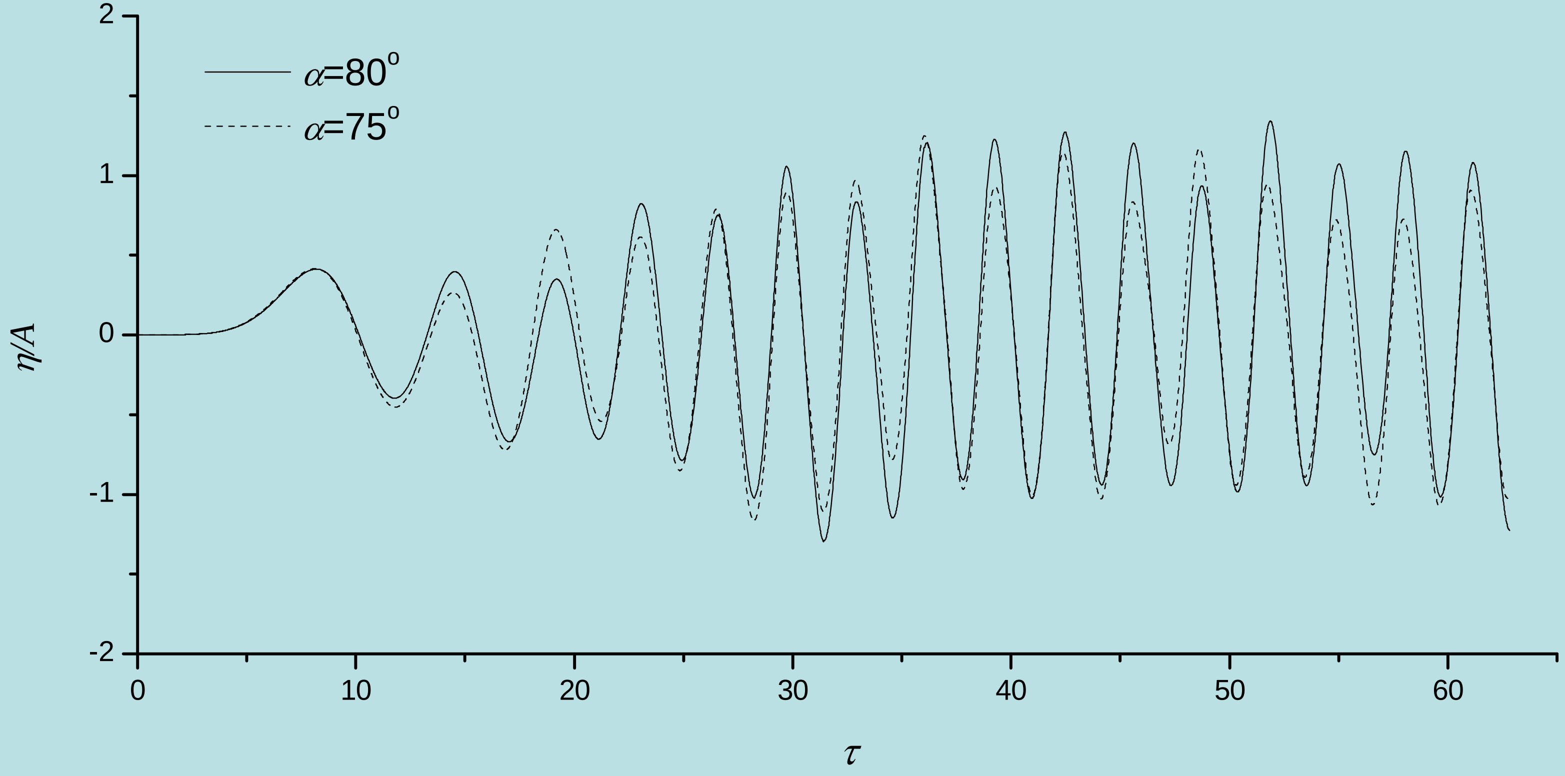


Sketch of a 3-D tank

$(L = 12h, B = 0.72h, L_{wc} = 7h, r = 0.1416h, X = -A\sin\omega t)$

Using unstructured mesh…

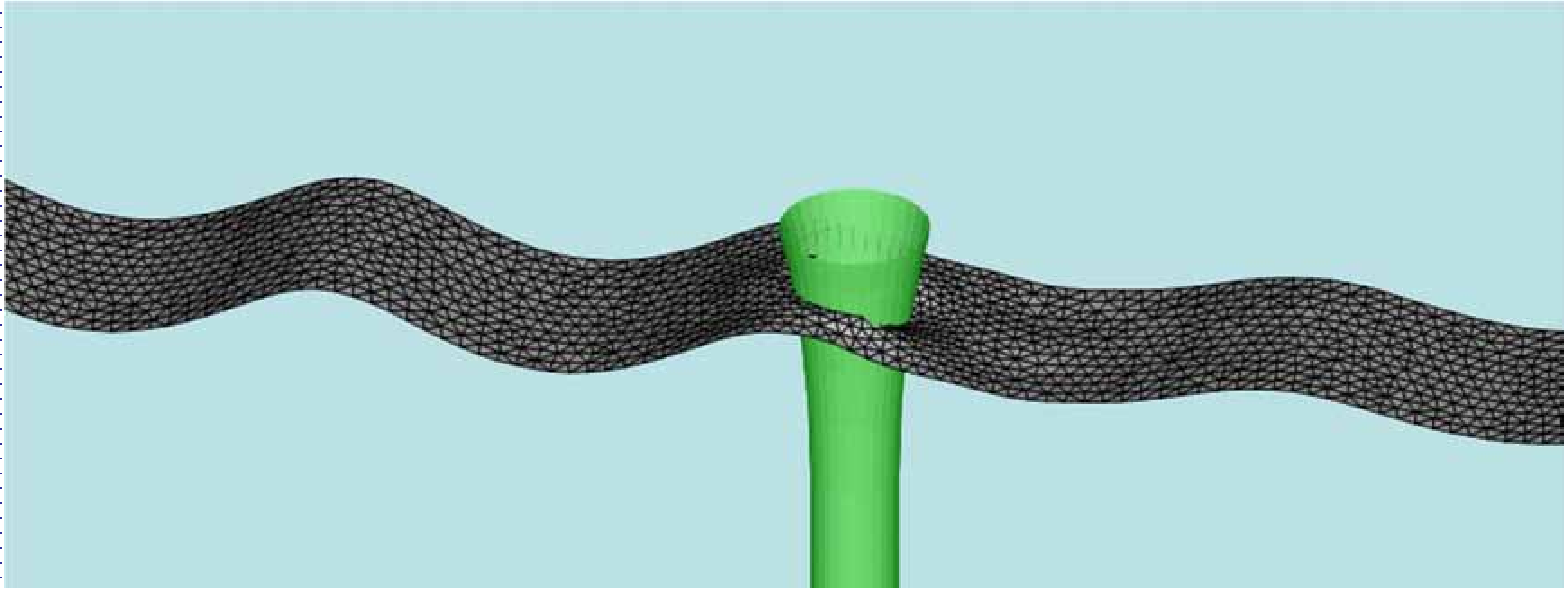
- 3-D wave-making problem for non-wall-sided cylinders (Wang, Wu & Drake, Ocean Eng. 2007)



Comparison of wave histories for two bottom mounted cylinders at A/h=0.04 ($\bar{\omega}$=2,x=L_{wc}-2r)

Using unstructured mesh…

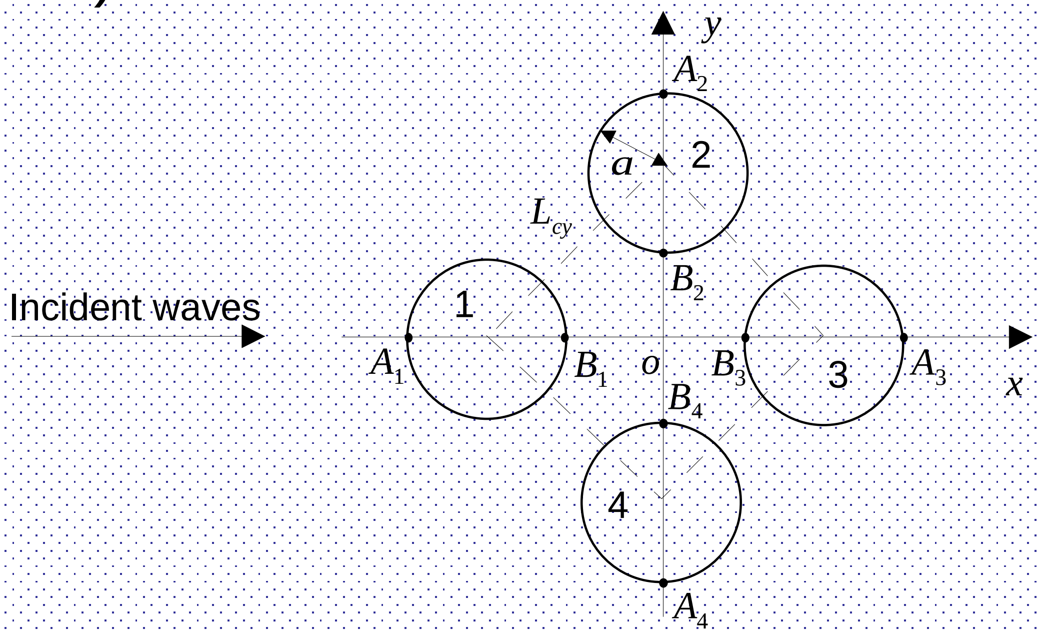
- 3-D wave-making problem for non-wall-sided cylinders (Wang, Wu & Drake, Ocean Eng. 2007)



Free surface around the cylinder ($\alpha=75^o$)

Using unstructured mesh…

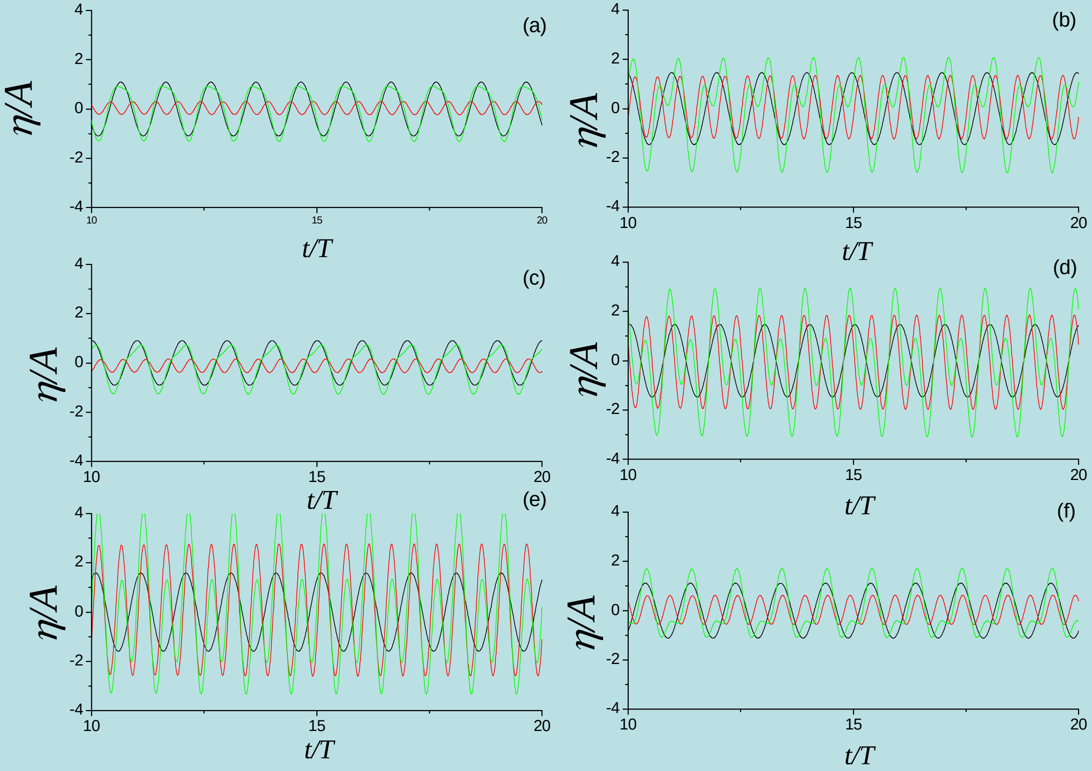
- Four-cylinder cases at the second order trapped mode frequency (Wang & Wu 2006, J. Fluids & Strucs.)



Four-cylinder cases

Using unstructured mesh…

- Four-cylinder cases at the second order trapped mode frequency (Wang & Wu 2006, J. Fluids & Strucs.)



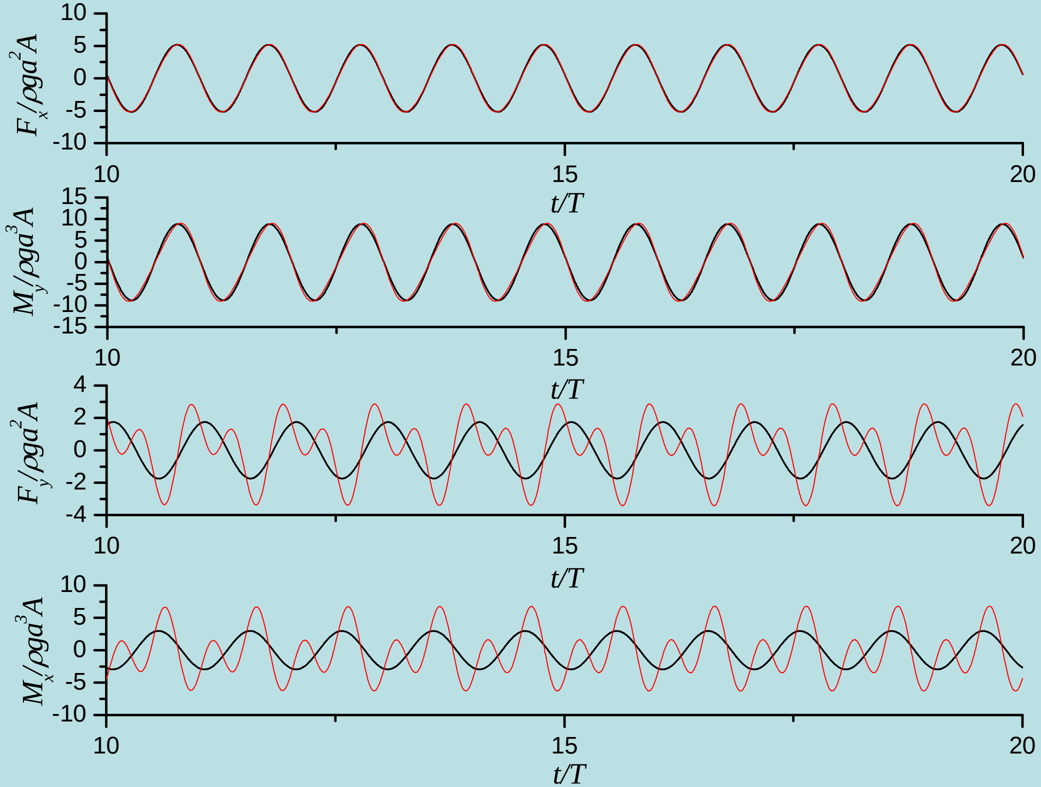
Wave histories for cylinders one, two and three at k_0a=0.468

(a) A_1; (b) B_1; (c) A_2; (d) B_2; (e) B_3; (f) A_3

linear; second order; linear plus second order

Using unstructured mesh…

- Four-cylinder cases at the second order trapped mode frequency(Wang & Wu 2006, J. Fluids & Strucs.)



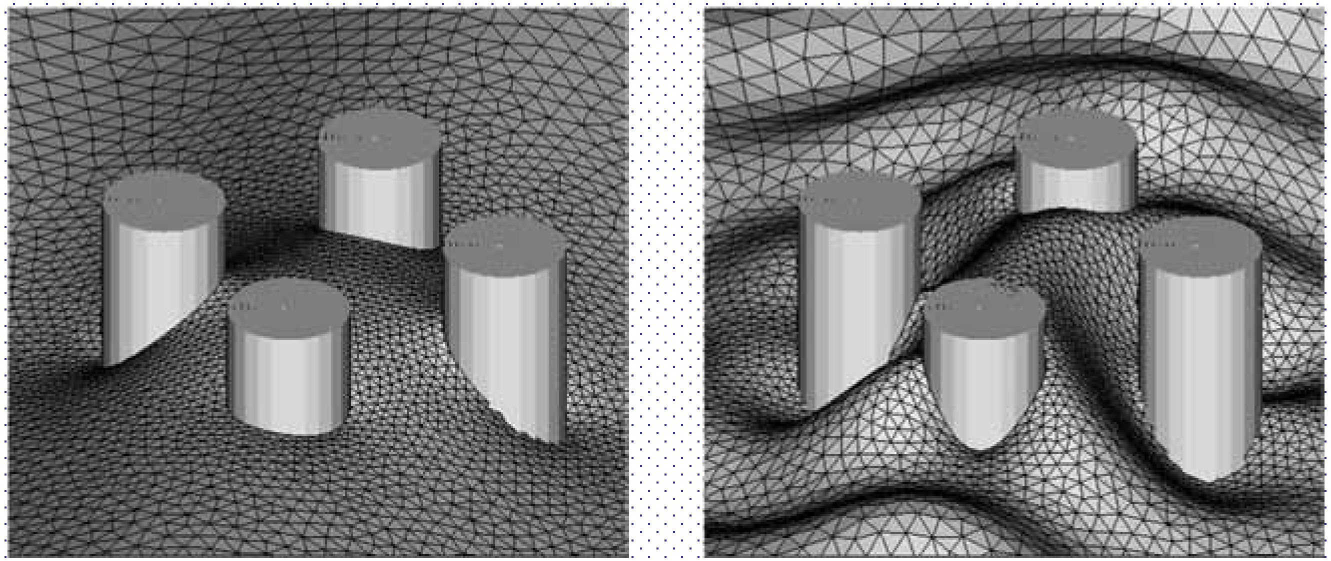
Histories of force and moment on cylinder two at $k_0 a$=0.468

—— linear; —— linear plus second order

Using unstructured mesh…

- Four-cylinder cases at the second order trapped mode frequency (Wang & Wu 2006, J. Fluids & Strucs.)

(a) (b)

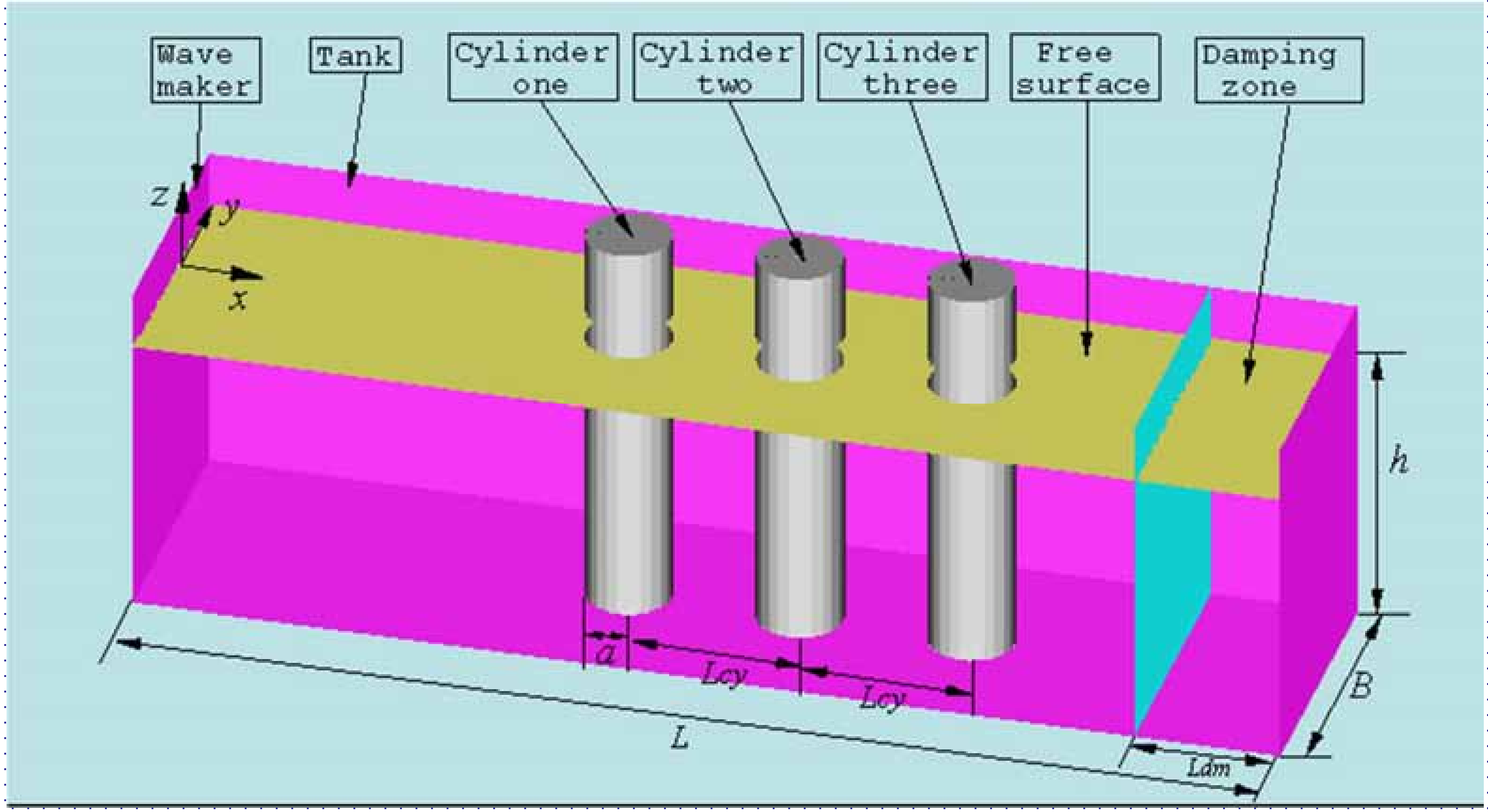


Wave profiles at $t=16T$

(a) linear; (b) linear plus second order

Using unstructured mesh…

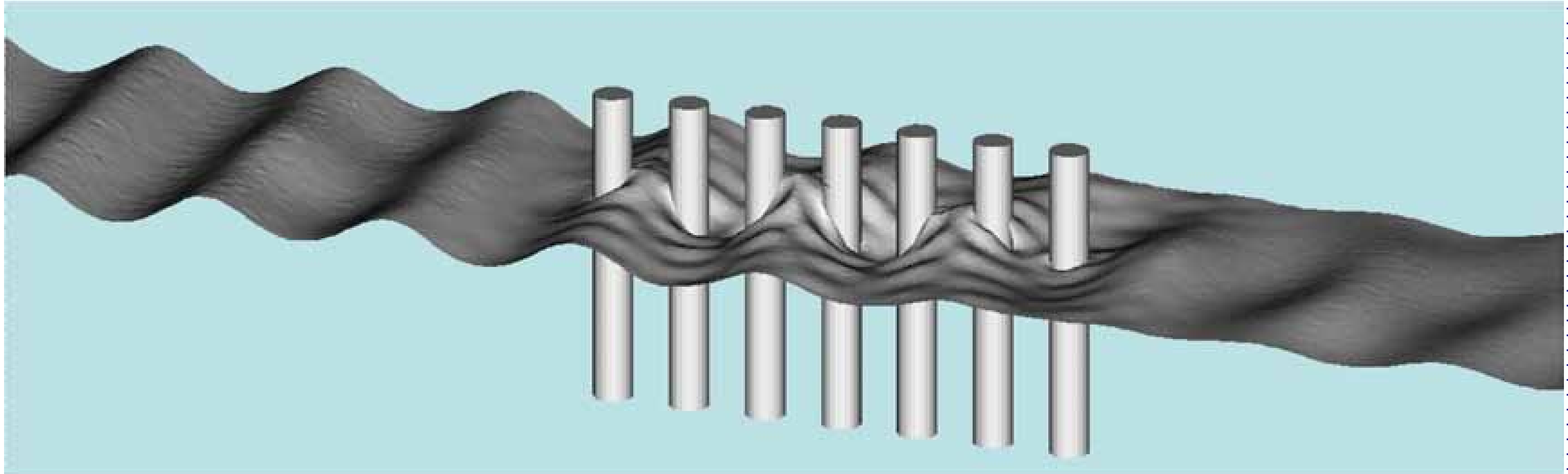
- 3-D wave-making problem for multiple cylinders (Wang & Wu 2010, Ocean Eng)



Sketch of the 3-D tank

Using unstructured mesh…

- 3-D wave-making problem for multiple cylinders.

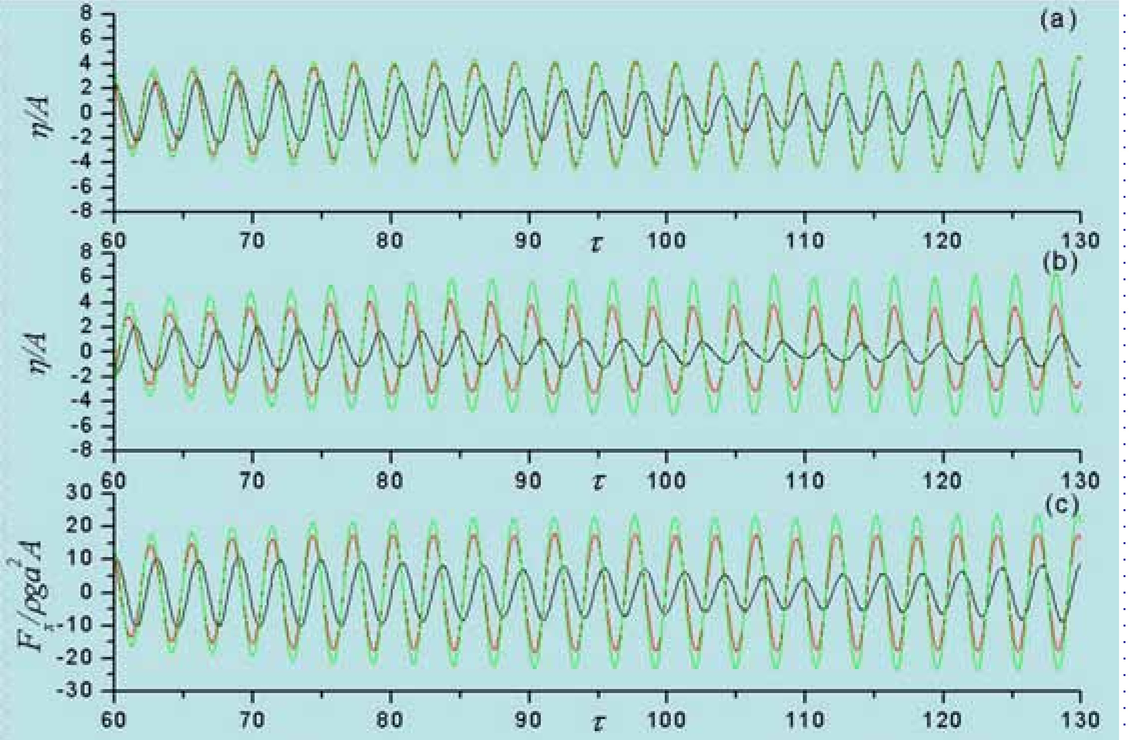


Wave profile at T=127.8 and A=0.02h

(Seven cylinder cases , r=0.1416, B=20r, h=1.0, L=18)

Using unstructured mesh…

- 3-D wave-making problem for multiple cylinders



The effect of tank wall on waves and force for cylinder four in seven-cylinder cases at A=0.005h and $\overline{\omega}$=2.15

—— B=5a; —— B=10a; —— B=20a

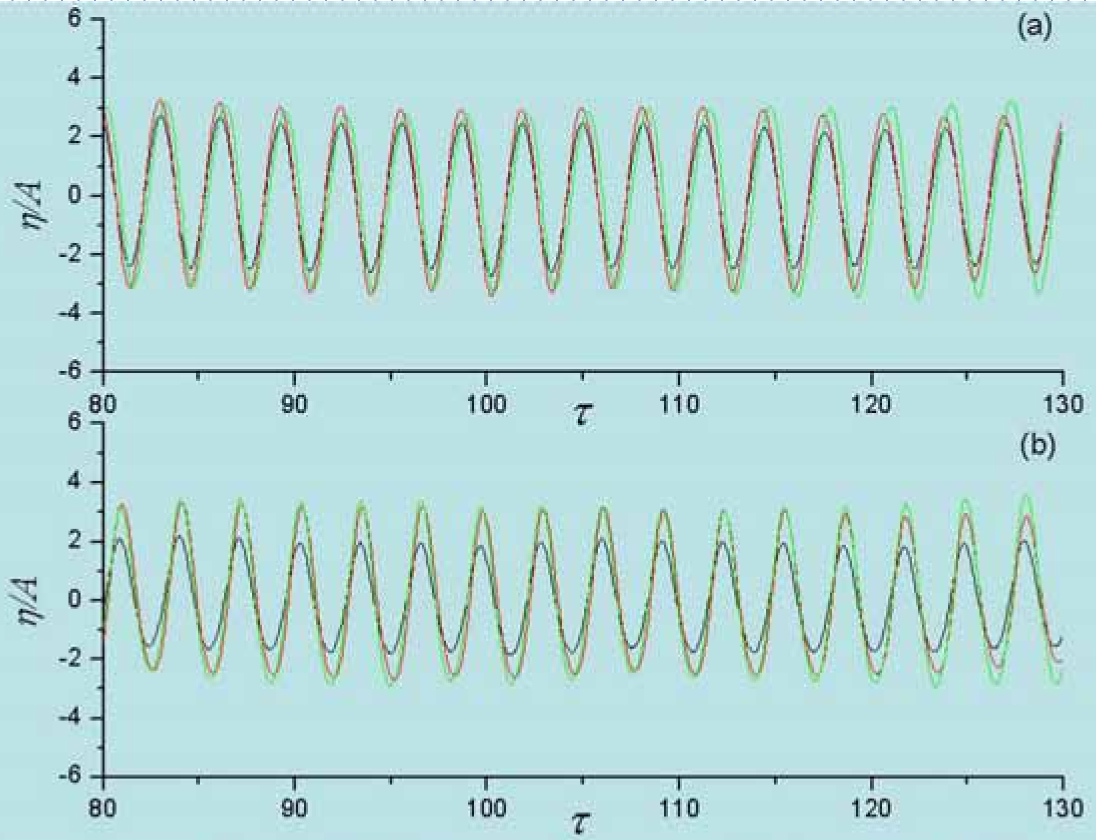
(a) waves at the front; (b) waves at the back; (c) force in x-direction

$\overline{\omega} = 2.15$ is close to the trapped mode frequency

(Seven cylinder cases, r=0.1416,B=20r, h=1.0,L=18)

Using unstructured mesh…

- 3-D wave-making problem for multiple cylinders



Comaparison of waves in single-,three- and seven-cylinder cases at $A=0.005h$ and $\bar{\omega}=2.0$

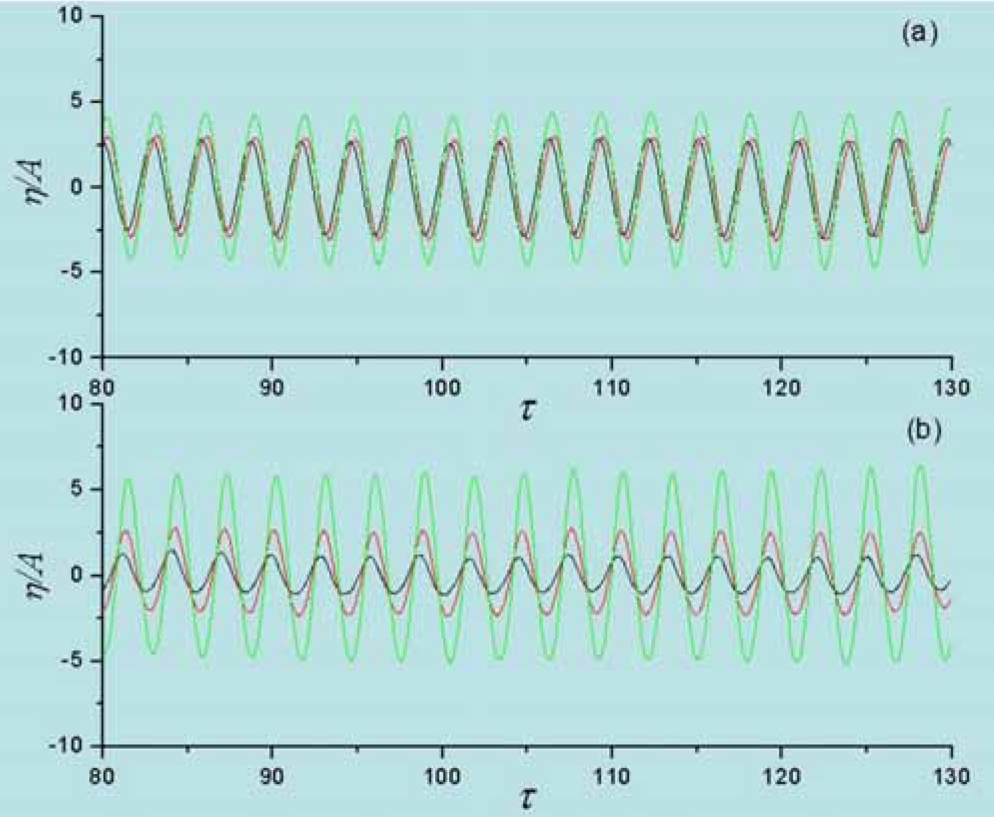
—— single-cylinder case; —— three-cylinder case; —— seven-cylinder case

(a) waves at the front of cylinder four; (b) waves at the back of cylinder four

(Seven cylinder cases, r=0.1416,B=20r, h=1.0,L=18)

Using unstructured mesh…

- 3-D wave-making problem for multiple cylinders



Comaparison of waves in single-,three- and seven-cylinder cases at A=0.005h and $\bar{\omega}$=2.15

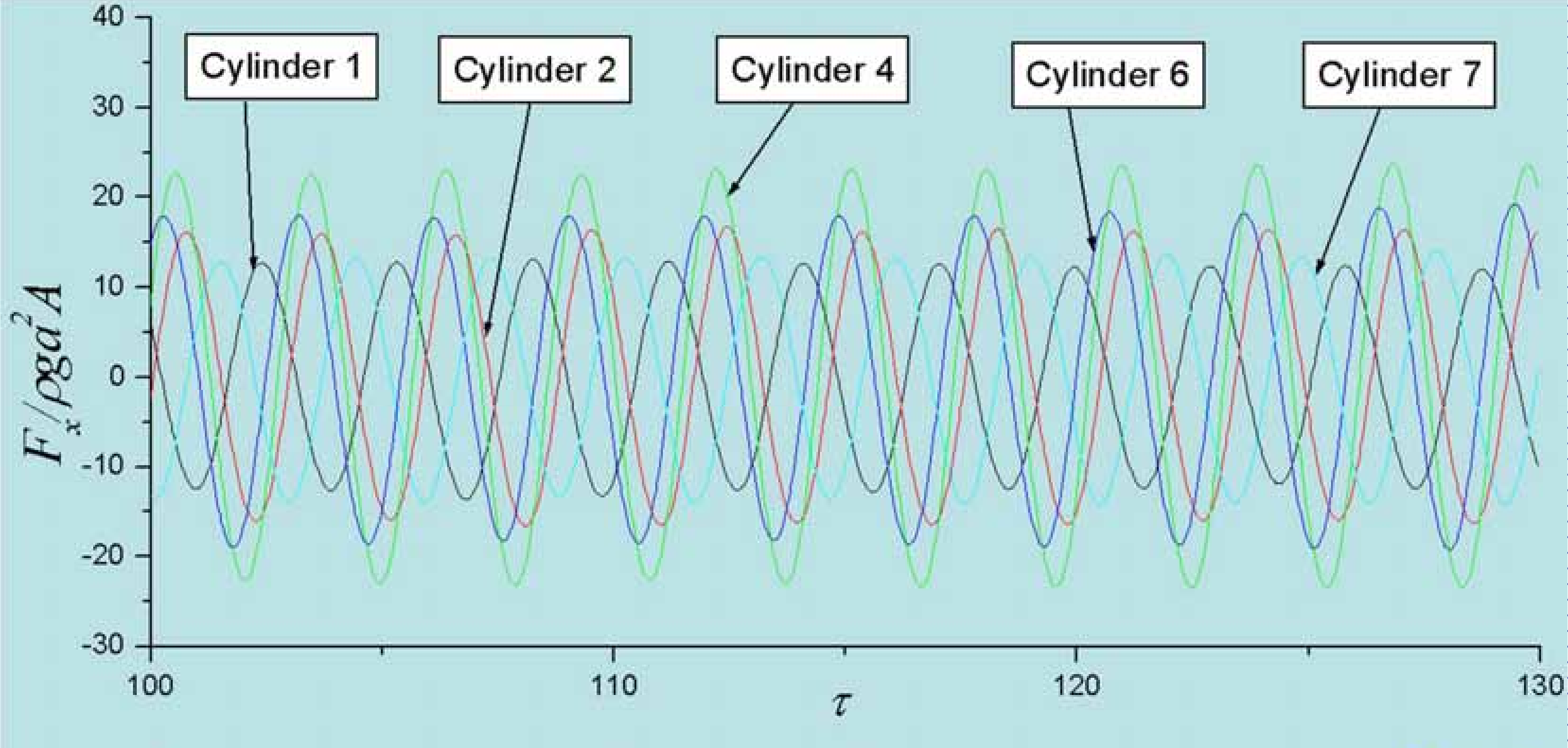
single-cylinder case; three-cylinder case; seven-cylinder case

(a) waves at the front of cylinder four; (b) waves at the back of cylinder four

(Seven cylinder cases, r=0.1416,B=20r, h=1.0,L=18)

Using unstructured mesh…

- 3-D wave-making problem for multiple cylinders



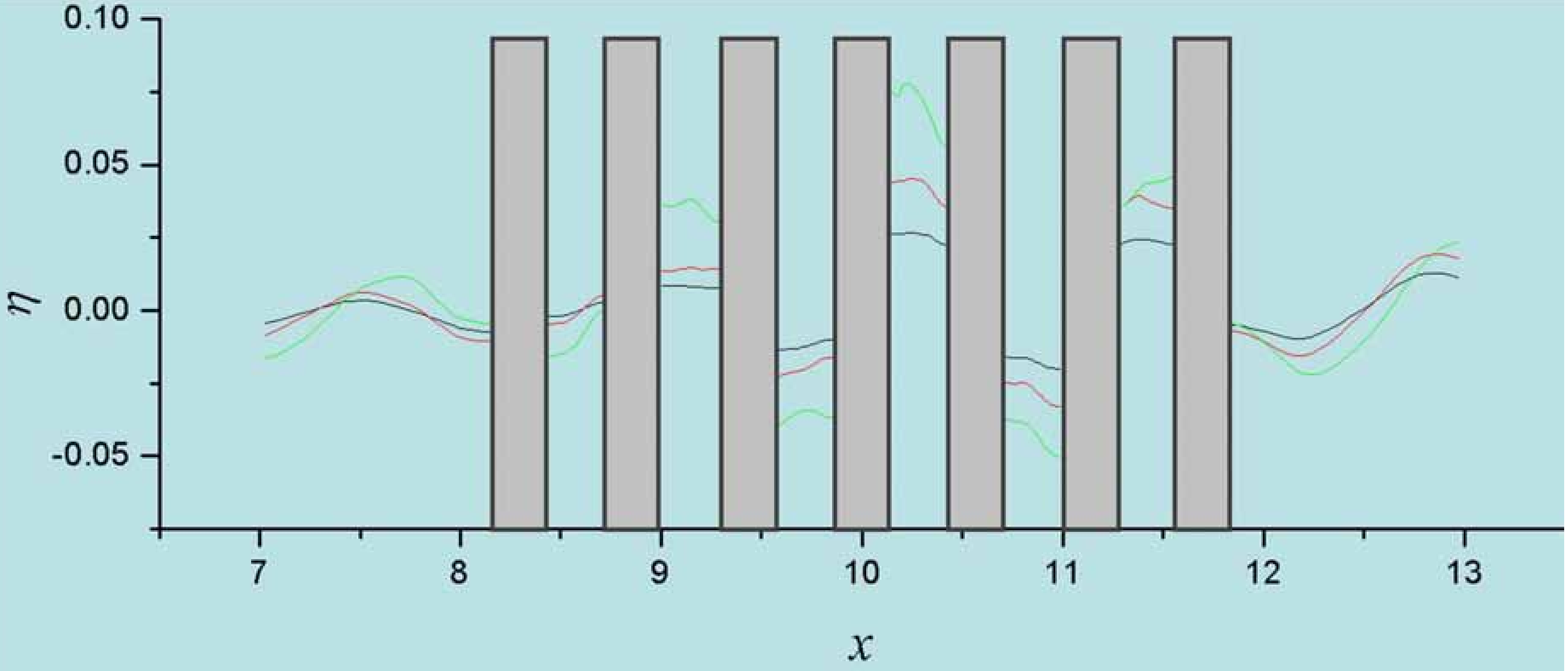
Forces on cylinders in the seven-cylinder case at A=0.005h and $\bar{\omega}$=2.15

The maximum force is on the middle cylinder (cylinder 4)

(Seven cylinder cases, r=0.1416,B=20r, h=1.0,L=18)

Using unstructured mesh…

- 3-D wave-making problem for multiple cylinders



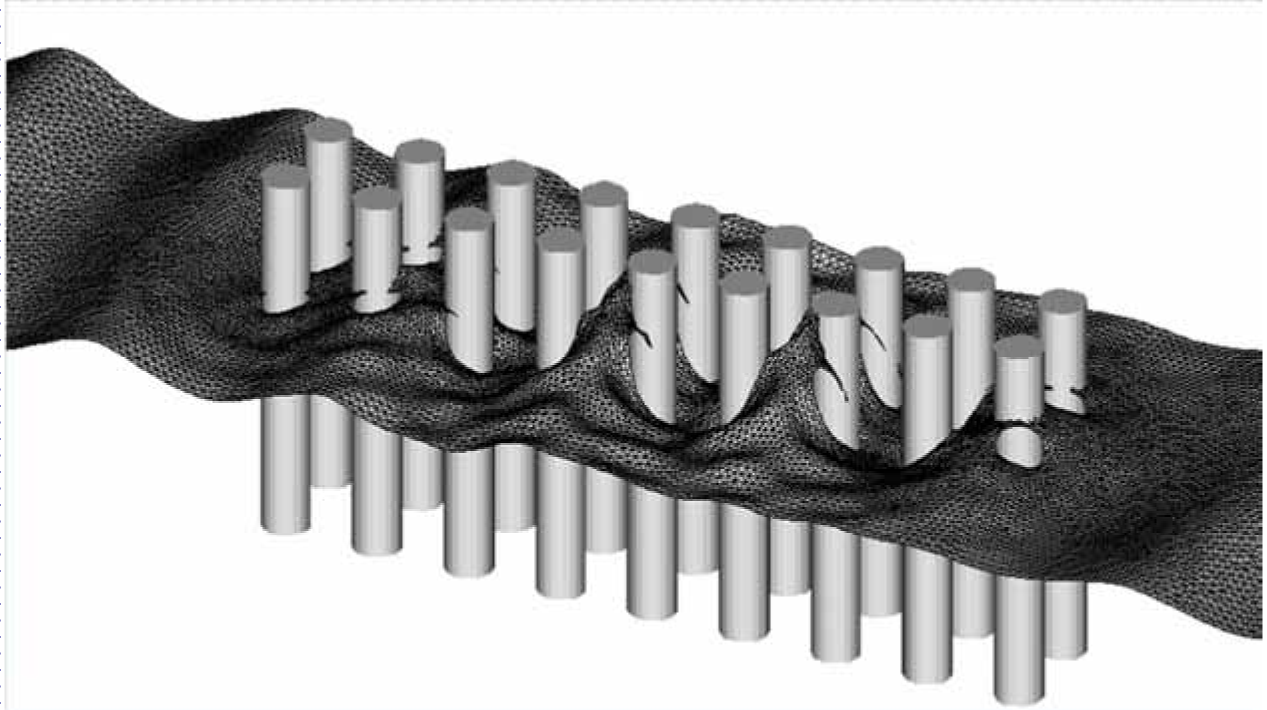
Wave profiles along y=0 at τ=127.9 in seven-cylinder cases at $\bar{\omega}$=2.15

A=0.005h; A=0.01h; A=0.02h

(Seven cylinder cases, r=0.1416, B=20r, h=1.0, L=18)

Using unstructured mesh…

- 3-D wave-making problem for multiple cylinders (Wang & Wu 2010, Ocean Eng)



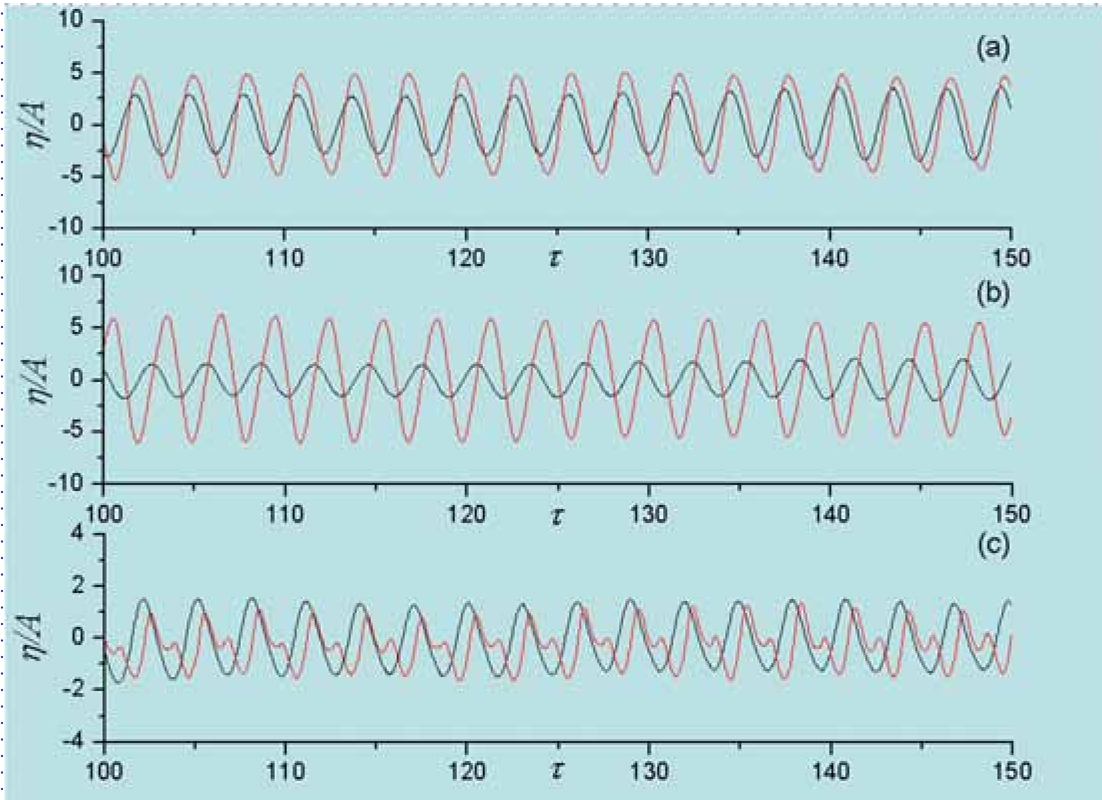
Wave profile at T=137.63 and A=0.02h ($\overline{\omega} = 2.1057$)

$\overline{\omega} = 2.1057$ is the trapped mode frequency

(Eighteen cylinder cases, r=0.1416,B=20r)

Using unstructured mesh…

- 3-D wave-making problem for multiple cylinders (Wang & Wu 2010, Ocean Eng)

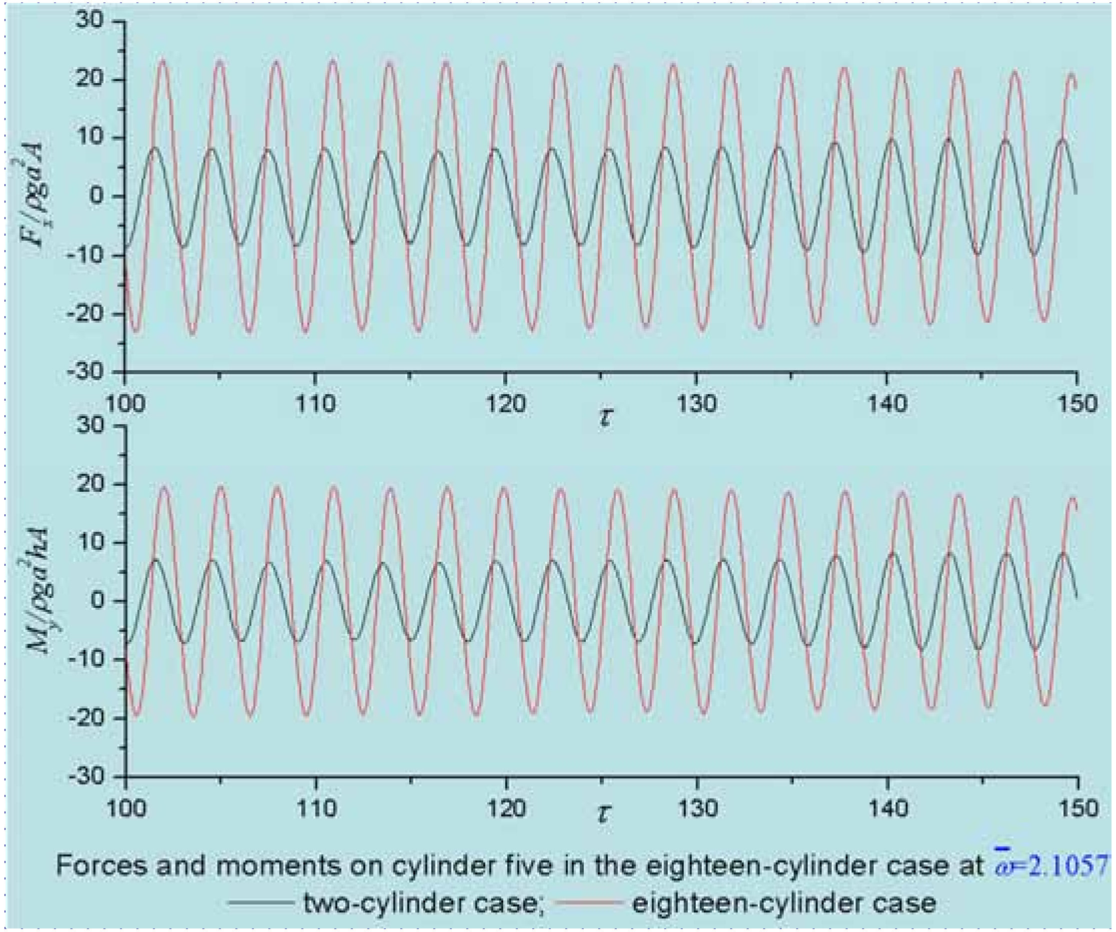


Comaparison of waves in two- and eighteen-cylinder cases at A=0.005h and $\bar{\omega}$=2.1057
—— two-cylinder case; —— eighteen-cylinder case
(a) front of cylinder five; (b) back of cylinder five; (c) the interior side of cylinder five.

(Eighteen cylinder cases, r=0.1416,B=20r)

Using unstructured mesh…

- 3-D wave-making problem for multiple cylinders (Wang & Wu 2010, Ocean Eng)

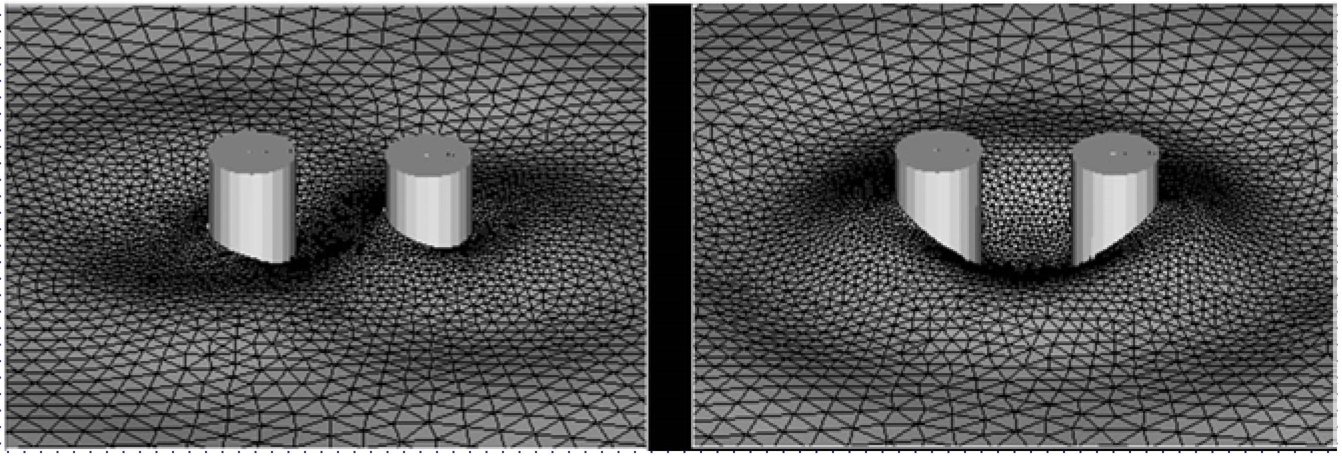


(Eighteen cylinder cases, r=0.1416,B=20r)

Using unstructured mesh…

- 3-D wave radiation by multiple cylinders: two-cylinder cases in horizontal motions

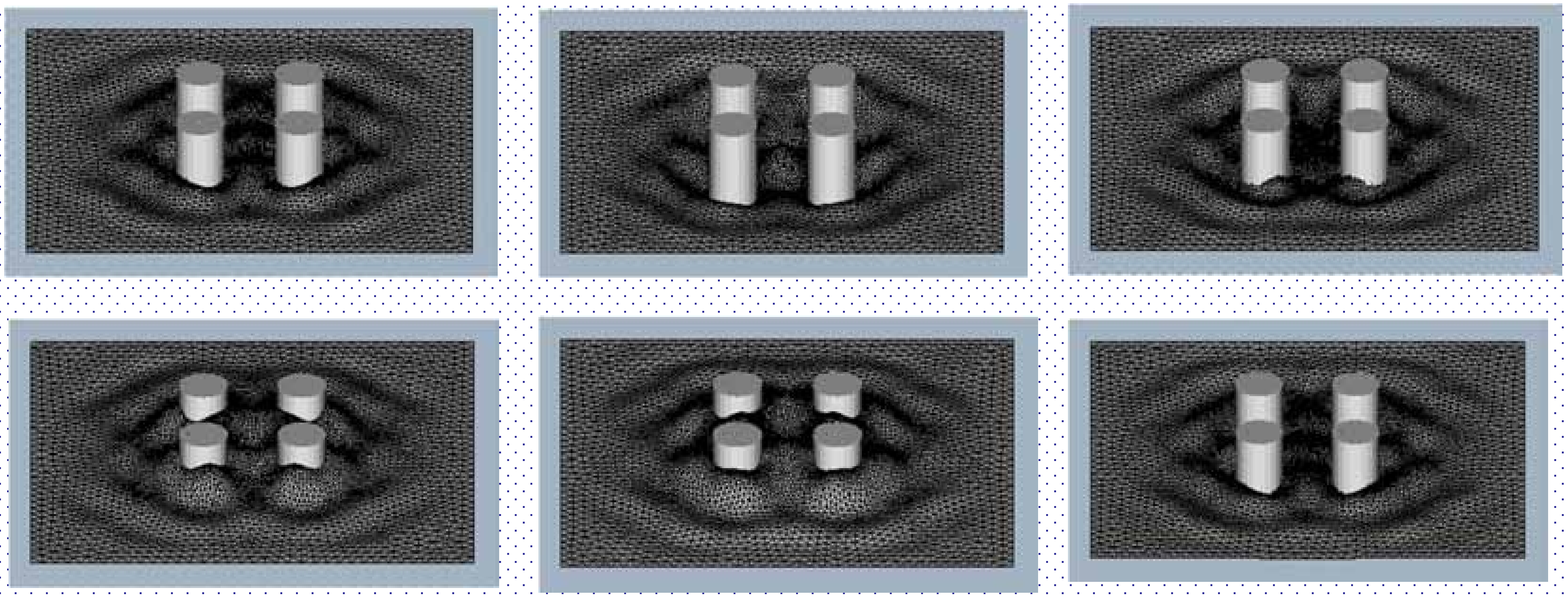
$$(a)\ X_1 = X_2 = A\sin\omega t \text{ and } (b)\ X_1 = A\sin\omega t, X_2 = -X_1$$



$$(h = 3a, d = 1.5a, A = 0.06a, L_{cy} = 4a, ka = 1.0)$$

Using unstructured mesh…

- 3-D wave radiation by multiple cylinders: four-cylinder cases in vertical motions $X = A\sin\omega t$



Meshes around cylinders at t=8T, 8.2T, 8.4T, 8.6T, 8.8T, 9T

$(h = 3a, d = 1.5a, A = 0.6a, L_{cy} = 4a, ka = 1.0)$

Summary

- The finite element method is efficient in simulations of nonlinear wave-structure interactions;
- Both structured and unstructured meshes can be used in the simulations. The former is more stable in the simulation and the and the latter is more suitable for complex domains;
- Enhanced interactions between multiple structures are strong at the resonant or the nearly trapped mode frequency.
- The waves and forces have strong nonlinear features at the first and second order resonant or the nearly trapped mode frequency.;
- Methods to calculate the velocity still need further study in both 2-D and 3-D cases when using unstructured meshes.
- It is still a big challenge to use fully 3-D unstructured meshes in fully nonlinear wave simulations.

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Thank you!

NOTE 3: FINITE ELEMENT METHOD

The physical concept of finite element method in fluid mechanics is different from that in structural mechanics. In the latter case, a structure is divided into small elements. On each element, the external force is balanced by the stress. In the former case, although the fluid is also divided into small elements, the governing equations are not established by a similar argument. The concept of the finite element method in this case is rather mathematical.

1. Governing equation

Let us consider a two dimensional case. We seek the solution of the following equations

$$\nabla^2\phi = 0 \tag{1}$$

in the fluid domain R and

$$\frac{\partial\phi}{\partial n} = Un_x \tag{2}$$

on the body surface S_0 (the definitions of various parameters here are the same as in notes 1 and 2). The fluid domain can be divided into many elements with n nodes. The potential may be written in terms of the finite element shape function $N_j(x,y)$, or

$$\phi_a = \sum_{j=1}^{n} \phi_j N_j(x,y) \tag{3}$$

where ϕ_j are the nodal values of the potentials. Equation (3) is clearly an approximation as indicated by the subscript a. It does not satisfy equations (1) and (2) exactly. The real task is to find a set of ϕ_j so that the error is minimized for a given n.

Substituting equation (3) into (1), we have

$$\nabla^2\phi_a = \varepsilon(x,y) \tag{4}$$

Ideally, we wish that the error ε would be zero. In practice, this is not possible. Thus, we use the following equation

$$\int_R \varepsilon(x,y)N_i \mathrm{d}R = 0 \qquad i = 1,2,\ldots,n \tag{5}$$

to make ε as small as possible. Substituting equation (4) into (5), we have

$$\int_R \nabla^2\phi_a N_i \mathrm{d}R = 0 \tag{6}$$

This equation can be further written as

$$0=\int_R \nabla^2\phi_a N_i \mathrm{d}R$$

$$=\int_R\left[\frac{\partial^2\phi_a}{\partial x^2}N_i+\frac{\partial^2\phi_a}{\partial y^2}N_i\right]\mathrm{d}R$$

$$=\int_R\left[\frac{\partial}{\partial x}(\frac{\partial\phi_a}{\partial x}N_i)+\frac{\partial}{\partial y}(\frac{\partial\phi_a}{\partial y})N_i-\frac{\partial\phi_a}{\partial x}\frac{\partial N_i}{\partial x}-\frac{\partial\phi_a}{\partial y}\frac{\partial N_i}{\partial y}\right]\mathrm{d}R$$

Applying Gauss's theorem (equation (1) of note 1) to the first two terms of the last equation, we obtain

$$0=\int_{S_0}\left[\frac{\partial\phi_a}{\partial x}N_in_x+\frac{\partial\phi_a}{\partial y}N_in_y\right]dS-\int_R\left[\frac{\partial\phi_a}{\partial x}\frac{\partial N_i}{\partial x}+\frac{\partial\phi_a}{\partial y}\frac{\partial N_i}{\partial y}\right]\mathrm{d}R$$

We further use

$$\frac{\partial\phi_a}{\partial n}=\frac{\partial\phi_a}{\partial x}n_x+\frac{\partial\phi_a}{\partial y}n_y$$

and impose equation (2) on ϕ_a. The above equation becomes

$$\int_R\left[\frac{\partial\phi_a}{\partial x}\frac{\partial N_i}{\partial x}+\frac{\partial\phi_a}{\partial y}\frac{\partial N_i}{\partial y}\right]\mathrm{d}R=U\int_{S_0}n_xN_i\mathrm{d}S \tag{7}$$

Substituting equation (3) into (7), we have

$$\sum_{j=1}^{n}\phi_j\int_R\left[\frac{\partial N_j}{\partial x}\frac{\partial N_i}{\partial x}+\frac{\partial N_j}{\partial y}\frac{\partial N_i}{\partial y}\right]\mathrm{d}R=U\int_{S_0}n_xN_i\mathrm{d}S \tag{8}$$

In matrix form, this becomes

$$[A][\phi]=[B] \tag{9}$$

where A is a square matrix with coefficients as

$$A(i,j)=\int_R\left[\frac{\partial N_j}{\partial x}\frac{\partial N_i}{\partial x}+\frac{\partial N_j}{\partial y}\frac{\partial N_i}{\partial y}\right]\mathrm{d}R \tag{10a}$$

B is a column with coefficients as

$$B(i)=U\int_{S_0}n_xN_i\mathrm{d}S \tag{10b}$$

and ϕ is a column which contains the unknown ϕ_j.

It is evident now that the remaining task is to calculate A and B for a given shape function N_j and to solve equation (9).

2. Shape function

There are variety of choices of shape functions. As a demonstration, we choose the linear shape function together with the triangular element (see figure 1a), which is defined as

$$N_j(x,y) = (a_j + b_j x + c_j y)/2\Delta \tag{11}$$

where

$$a_1 = x_2y_3 - x_3y_2 \qquad a_2 = x_3y_1 - x_1y_3 \qquad a_3 = x_1y_2 - x_2y_1 \tag{12a}$$

$$b_1 = y_2 - y_3 \qquad b_2 = y_3 - y_1 \qquad b_3 = y_1 - y_2 \tag{12b}$$

$$c_1 = x_3 - x_2 \qquad c_2 = x_1 - y_2 \qquad c_3 = x_2 - x_1 \tag{12c}$$

and $\Delta = (a_1+a_2+a_3)/2$ is the area of the element. It is easy to confirm that the shape function has the following properties

$$\sum_{j=1}^{3} N_j(x,y) = 1 \tag{13a}$$

$$\begin{cases} N_j(x_i,y_i) = 1 & i = j \\ N_j(x_i,y_i) = 0 & i \neq j \end{cases} \tag{13b}$$

3 Local matrix and global matrix

One distinct feature of the finite element method is that the shape function discussed in equations (11) and (12) correspond locally to a particular element while equation (9) is in the global system. The procedure to solve the problem is to consider element by element first. The global matrix is then obtained by assembling the local results for each element.

We consider a single element in figure 1*a*. Substituting equation (11) into (10a), we have

$$A^1(i,j) = \int_R (b_ib_j + c_ic_j)/4\Delta^2 \mathrm{d}R$$

$$= (b_ib_j + c_ic_j)/4\Delta^2 \int_R \mathrm{d}R$$

where subscript 1 indicates that the coefficients correspond to element 1. The result of the integration in above equation is clearly the area of the element. Thus

$$A^1(i,j) = (b_ib_j + c_ic_j)/4\Delta \tag{14}$$

When there is only one element, the global matrix is the same the local matrix, or

$$[A] = \begin{bmatrix} A^1(1,1) & A^1(1,2) & A^1(1,3) \\ A^1(2,1) & A^1(2,2) & A^1(2,3) \\ A^1(3,1) & A^1(3,2) & A^1(3,3) \end{bmatrix} \tag{15}$$

We now add one more element into the problem as shown in figure 1*b*. The numbers with a circle correspond to the global system while those without correspond to the local system. The global matrix becomes

$$[A] = \begin{bmatrix} A^1(1,1) + A^2(3,3) & A^1(1,2) + A^2(3,2) & A^1(1,3) & A^2(3,1) \\ A^1(2,1) + A^2(2,3) & A^1(2,2) + A^2(2,2) & A^1(2,3) & A^2(2,1) \\ A^1(3,1) & A^1(3,2) & A^1(3,3) & 0 \\ A^2(1,3) & A^2(1,2) & 0 & A^2(4,4) \end{bmatrix} \quad (16)$$

4. Exercise

Find the global matrix when one more element is added into the problem (see figure 1c)

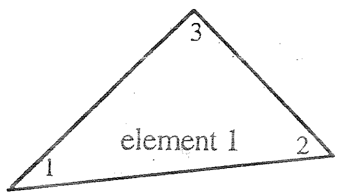

Figure 1a

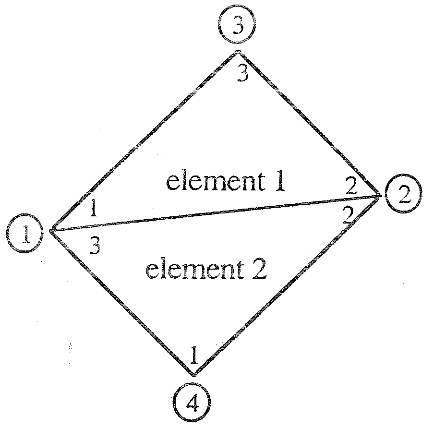

Figure 1b

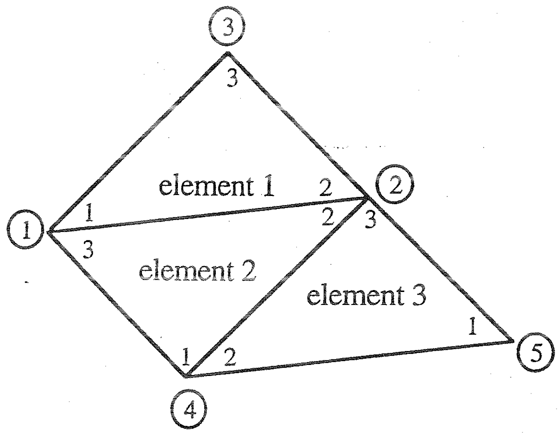

Figure 1c